Dependence structures and the pricing of CDO's.

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Content

Motivation.

Is there a model capable of matching **any given** set of CDSs (or CDOs) prices?

Goals

1. Identify the most relevant features of dependence for CDS.

2. Provide a simple multivariate distribution capable of matching those features of dependence.

Standard Features of Dependence

- Pearson Correlation.

- Spearman's rho Correlation.

- Kendall's tau Correlation.

- Tail Dependence.

These features seem to be either **irrelevant** or **not enough** in the context of describing relationships for CDS prices.

Current Families of Models

- Copula-Factor (Hull-White 2004, Laurent-Gregory 2003).
- Intensity-Factor (Duffie-Garleanu 2001, Hurd-Kuznetsov 2004).
- Structural Model (Escobar-Seco 2004, Hull-White 2005).

These models could be more effective (for matching CDSs and CDOs prices) if the targeted features of dependence were clear.

Notation.

N. Bonds in the Pool.

τ_i. Time of default, Bond i.

T. Maturity Time.

$S^k(T) = \sum^{N_k} Q(\tau_{i_1} < T, .., \tau_{i_k} < T, \tau_{i_{k+1}} > T, .., \tau_{i_N} > T)$, Probability of exactly k defaults by time T. $N_k = \frac{N!}{(N-k)!\cdot k!}$.

$L_i = L$. Notional.

$\alpha_i = \alpha$. Probability of default, Bond i.

$\delta_i = 0$. Recovery rate.

r. Interest rate.

M. Tranches in the CDO.

$(d_{K-1}, d_K]$. Range of defaults for tranche K.

Simple Pricing Schemes.

Price k to default CDS:

$$\Psi_k = e^{-r \cdot T} \cdot L \cdot k \cdot S^k(T, ..., T) \quad (1)$$

Price of Tranche k, CDO:

$$\Upsilon_k \quad = \sum_{i=d_{k-1}}^{d_k} e^{-r \cdot T} \cdot L \cdot (i - d_{k-1}) \cdot S^i(T, ..., T) \quad (2)$$

Remark I: CDS and CDOs prices are matched if S^k, $k = 1, ..., N$ are matched.

Goal 1. S^k as the Key Property of Dependence.

Constrains for S^k, $k = 1, .., N$

$$\sum_{k=0}^{N} S^k(T, ..., T) = 1 \tag{3}$$

$$S^1(T, ..., T) + \sum_{k=1}^{N-1} (k+1) \cdot S^{k+1}(T, ..., T) = N \cdot \alpha \tag{4}$$

$$0 \leq S^k(T, ..., T) \leq \min\left(\alpha \frac{N}{k}, 1\right); k = 1, ..., N \tag{5}$$

Remark II: These constrains show the feasible set of values for S^k.

Goal 2: Mixture Model (MM)

- Define:

$$q^{i_1,\dots,i_k}(\tau_1,...,\tau_N) = q_d(\tau_{i_1}) \cdot ... \cdot q_d(\tau_{i_k}) \cdot q_{nd}(\tau_{i_{k+1}}) \cdot ... \cdot q_{nd}(\tau_{i_N}), \quad (6)$$

  where q_d and q_{nd} are nonzero densities in $[0,T]$, $[T,\infty)$ respectively.

- Denote $p_{i_1,\dots,i_k}$: probability of default from firms $(i_1,...,i_k)$ and $p^{(0)}$: probability of no default. Assume $p_{i_1,\dots,i_k} = p^{(k)}$, $\forall(i_1,...,i_k)$.

Multivariate density of $\tau = (\tau_1,...,\tau_N)$:

$$q(\tau) = p^{(0)} \cdot q^0(\tau) + p^{(1)} \cdot \sum_{i=1}^{N} q^i(\tau) + p^{(2)} \cdot \sum_{i_1,i_2}^{N_2} q^{i_1,i_2}(\tau) + ... + p^{(N)} \cdot q^{1,\dots,N}(\tau). \quad (7)$$

Remarks III:

1. Evaluating $q(\tau)$ at a point is equivalent to evaluating a single density $q^{i_1,\ldots,i_k}(\tau)$, all other densities are zero.

2. $p^{(k)}$, $k = 1, ...N$, are the parameters of this model.

3. The marginal density of τ_i is: $q(\tau_i) = \alpha \cdot q_d(\tau_i) + (1 - \alpha) \cdot q_{nd}(\tau_i)$.

4. The parameters of q_d, q_{nd} (if any) can be chosen to match marginal features, i.e. mean, volatility.

Remarks IV:

1. If Q_d and Q_{nd} are closed form then joint cumulative distribution $Q(\tau_{i_1}, ..., \tau_{i_k})$ have a closed form.

2. From construction of MM and $S^k = \sum_{i_1,...,i_k} p_{i_1,...,i_k}$ follows $p^{(k)} = \frac{S^k}{N_k}$ (easy estimation from CDS prices).

3. Any set of CDS prices $(\Psi_1, ..., \Psi_N)$ can be matched with a convenient set of parameters $(p^{(1)}, ..., p^{(N)})$.

Application of MM to CDOs.

- Notice the set of probabilities $p_{i_1,...,i_j}, j \in [d_{k-1}, d_k]$ are related to defaults in tranche k.

- Any set of tranche' prices $(\Upsilon_1, ..., \Upsilon_M)$ can be matched using a convenient set of M parameters $(p^{(1)}, ..., p^{(M)})$.

- I.e. Assume $p^{(j)} = p^{(k)}$, $\forall j \in [d_{k-1}, d_k]$, and take the parameters $p^{(k)}$ as follow:

$$p^{(k)} = \frac{\sum_{i=d_{k-1}}^{d_k} (i - d_{k-1}) \cdot S^i(T, ..., T)}{N_k \cdot \sum_{i=d_{k-1}}^{d_k} (i - d_{k-1})} \quad (8)$$

If $\widehat{S}^k = p^{(k)} \cdot N_k$ satisfy constrains 3-5 then p^k is the convenient set.

Empirical Results: Matching CDO's spreads

Setting:

$N = 100$	$M = 5$	$L = 1$	$\delta = 0$	$r = 0.05$	$T = 1$
$d_0 = 0$	$d_1 = 1$	$d_2 = 4$	$d_3 = 10$	$d_4 = 13$	$d_5 = 100$

Tranches' spread, bp (3 Factor Gaussian Model):

$Eq = 1250$	$Mz_1 = 847$	$Mz_2 = 410$	$Mz_3 = 217$	$Sn = 3.1$

MM parameters:

$p^{(1)} = 0.0235$	$p^{(2)} = 0.0081$	$p^{(3)} = 0.0040$	$p^{(4)} = 0.0043$	$p^{(5)} = 0.0003$

Tranches' spread, bp (Marshall-Olkin Copula Model):

$Eq = 1250$	$Mz_1 = 673$	$Mz_2 = 270$	$Mz_3 = 147$	$Sn = 18.1$

MM parameters:

$p^{(1)} = 0.0235$	$p^{(2)} = 0.0065$	$p^{(3)} = 0.0026$	$p^{(4)} = 0.0029$	$p^{(5)} = 0.0018$

Generalizations

Multivariate density of τ:

$$q(\tau) = p_0 \cdot q^0(\tau) + \sum_{i}^{N} p_i \cdot q^i(\tau) + \sum_{i_1,i_2}^{N_2} p_{i_1,i_2} \cdot q^{i_1,i_2}(\tau) + ... + p_{1,\ldots,N} \cdot q^{1,\ldots,N}(\tau) \tag{9}$$

where:

- The vector of weights $p_0, ..., p_{1,\ldots,N}$ is such that $p_{i_1,\ldots,i_k} > 0, \sum p_{i_1,\ldots,i_k} = 1$.
- $q^{i_1,\ldots,i_k}(\tau)$ is a multivariate density defined as before.

Cases

- Non-homogeneous notional:

  No change in the Mixture model. Notice that the price of a k to default CDS for notional $L_i, i = 1, ..., N$ would be:

$$= B(0,T) \cdot \sum_{i_1,...,i_k} (L_{i_1} + ... + L_{i_k}) \cdot p_{i_1,...,i_k}$$
$$= B(0,T) \cdot p^k \cdot \sum_{i_1,...,i_k} (L_{i_1} + ... + L_{i_k})$$

- Non-homogeneous credit ratings (Bonds AAA,AA,BBB, etc):

  The assumption $p_{i_1,...,i_k} = p^k$, $\forall (i_1, ..., i_k)$ fails, so non-constant $p_{i_1,...,i_k}$ should be estimated using system of equations derived from the equation $S^k = \sum_{i_1,...,i_k} p_{i_1,...,i_k}$ (Non-Symmetrical Mixture models).

Conclusions

- The S^k were studied as the most relevant feature of the dependence structure for the pricing of CDS and CDO.

- A closed form multivariate mixture model capable of matching any given set of CDSs or CDOs was presented.

- Joint probabilities of default as well as $S^k(T_1, ..., T_1), T_1 <> T$ and risk measures of portfolios of these variables have been computed in a fast an accurate manner.

- These results can be used for Risk Management as well as Derivative Pricing.

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