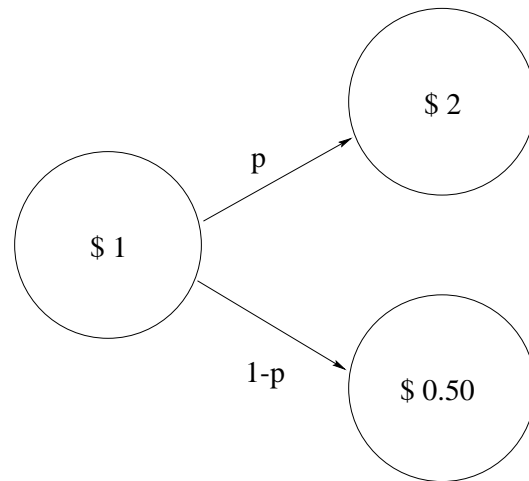
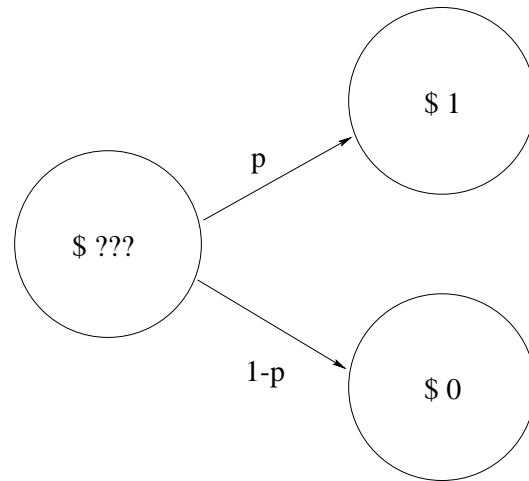


Example

Example. (Ignore interest rates). Call Option. Pays $f_0(S) = (S - \$1)_+$.



1-Period Stock Tree



Option Value = V

Assume $p = 50\%$. Is $V = 0.50$?

Answer: No!

$$V = 1/3\$.$$

If we borrow $\$1/3$, and we buy $\$2/3$ of S , we will exactly cover (or hedge) our payoff.

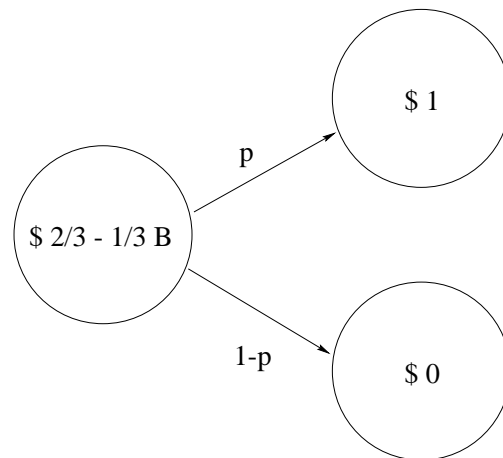
Since it costs $\$1/3$ to purchase this portfolio, the price should be the same.

Discounted Values

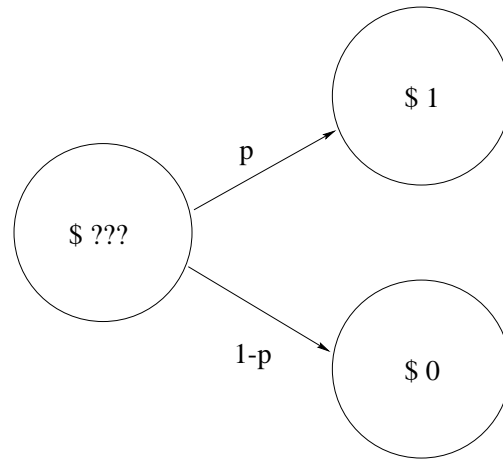
Time is money. Assume the existence of a bond with constant interest rate r .

We build the following portfolio Π :

$$\Pi = \frac{2}{3} \text{ Stock units} + \left(-\frac{1}{3}\right) \text{ bond}$$



Portfolio Values



Option Value

No matter what p is, absence of arbitrage implies

$$\begin{aligned}\text{Option Price} &= \frac{2}{3} - \frac{1}{3}B \\ &= \frac{2}{3} - \frac{1}{3}e^{-rT}.\end{aligned}$$

where T is the time to expiration and r is the (constant) interest rate.

Implied Probabilities

We can still achieve

$$\begin{aligned}\text{Option Price} &= \mathbb{E} \left(e^{-rT} f_0 \right) \\ &= p e^{-rT},\end{aligned}$$

by selecting

$$p = \frac{2}{3} e^{rT} - \frac{1}{3}.$$

In other words, we can construct a probability measure \mathbb{P} for the stock process, such that

$$\text{Option Price} = \mathbb{E}_{\mathbb{P}} \left(B_T^{-1} f_0 \right).$$

More generally, if we define the (arbitrage-free) price to equal the discounted pay-off

$$V = B_T^{-1} f_0,$$

then, there exists a measure \mathbb{P} under which V is a martingale: its value today is its expected future value.

Implied Market Data

Example: Assume the previous call option is sold for \$0.50.

$$\frac{2}{3} - \frac{1}{3} e^{-r} = 0.5.$$

Hence, the risk-free rate must equal

$$r = -\ln 2.$$

Incomplete markets

Example. Assume the stock valued at \$1 today, can be worth

$$S = \begin{cases} \$2 \\ \$1 \\ \$0.5 \end{cases}$$

after a year. How can we price the call option with strike 1?.

Two possibilities:

- Another derivative price is known
- We can re-balance our hedge once before maturity.

Pricing Theory

Pay-off matrix

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 0.5 \end{pmatrix}$$

The replicating strategy is given by

$$D \cdot \begin{pmatrix} x = \text{bond units} \\ y = \text{stock units} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Cost vector:

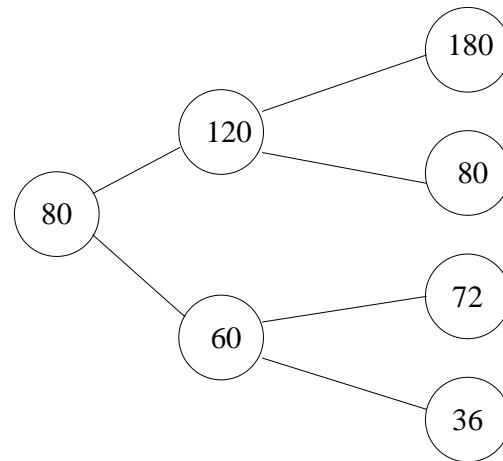
$$q = (0.9, 1).$$

Price:

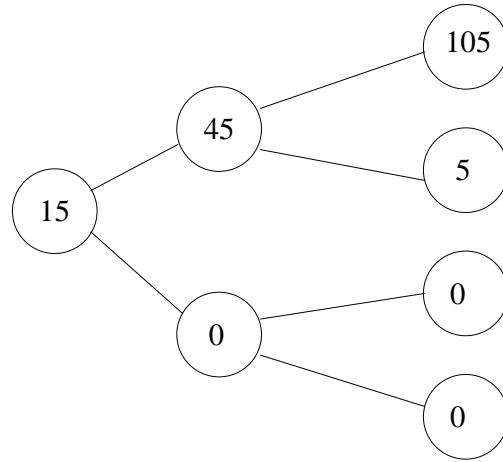
$$\begin{aligned} \text{Price} &= q \cdot x \\ &= q \cdot D^{-1} \cdot (\text{Pay-off vector}) \\ &= \text{Expected Pay-off} \end{aligned}$$

Multiperiod Pricing

Assume



A call option with strike \$75 can be priced as follows ($r=0$):



So its value today is \$15.

This is the arbitrage-free price. Implied probabilities can be obtained as usual.

Pricing Theory (One Period)

Implied probabilities can be obtained, not only from prices dictated by arbitrage arguments, but also from market prices.

The implications of this is that a probabilistic approach to pricing is more useful than might have seemed from the considerations above.

In this section we assume there is a probability space for the payoffs of N securities available for trading,

A security is characterized by its *cost* now, and its *payoff* after one unit of time.

The **cost of the i -th security**, $i = 1, \dots, N$, is q_i .

The **payoff** is given by the random variable $D_i(\omega)$.

The **expected payoff** of a security is $E(D_i(\omega))$.

A **portfolio** is a vector $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$, which represents the holdings of each security. θ_i can be positive or negative. If θ_i is positive, our position is

said to be long. If θ_i is negative, our position is said to be short.

The **payoff** of the portfolio θ is $\theta \cdot D(\omega)$.

A market is said **complete** if

$$\text{Span } \{ \theta \cdot D(\omega) , \theta \in \mathbb{R}^N \} = L^2(\mu).$$

and markets are usually assumed to be complete. In a complete market, for any payoff there is a portfolio with that payoff.

The **cost** of a portfolio θ is $q \cdot \theta$.

If a portfolio has nonzero cost, i.e. $q \cdot \theta \neq 0$, one defines its **return** to be

$$R_\theta(\omega) = \frac{\theta \cdot D(\omega)}{q \cdot \theta}.$$

In a real market, there are hedgers (people trying to minimize risk), speculators (people trying to maximize return) and arbitrageurs (people detecting market inefficiencies).

We say that there is an **arbitrage opportunity** if there is a portfolio θ such that

$$q \cdot \theta \leq 0, \text{ and } D \cdot \theta \geq 0 \quad a.e.,$$

and $D \cdot \theta > 0$ with non-zero probability.

The **Efficient Market Hypothesis (EMH)** states that there is no arbitrage and there are no transaction costs.

Theorem. (Riesz representation) *If p_i are linear functionals of the payoffs $L^2(\mu)$, then there exists a random variable $\pi(\omega)$ such that*

$$p \cdot \theta = E(\theta \pi \cdot D), \quad \text{all } \theta \in \mathbb{R}^N. \quad (1)$$

If markets are complete, π is unique. If there are no arbitrage opportunities, $\pi > 0$.

In the case that we consider the cost as that linear functional, we obtain that the cost of a portfolio is the expectation of its payoff with probabilistic weight

$\pi(\omega)$, which is called the **state–price deflator**. The name comes from the fact that

$$E(R_\theta \pi) = 1 \quad (2)$$

for all portfolios θ .

We always assume that $D_0(\omega)$ is constant for all $\omega \in \Omega$. This is a **savings account**.

A **riskless bond** is a portfolio θ_0 of constant payoff i.e. such that $\theta \cdot D(\omega) = \theta \cdot D(\omega')$ for all $\omega, \omega' \in \Omega$. It always exists: put $\theta = (1, 0, \dots, 0)$. Then from (2) we find

$$R^0 \equiv E(R_{\theta_0}) = \frac{1}{E(\pi)}.$$

The **riskless interest rate** is given by

$$r = -\frac{1}{T} \ln \mathbb{E}(R_{\theta_0}).$$

Theorem. *A price deflator exists if and only if there is no arbitrage.*

Proof. If a price deflator exists, then $\Pi(0) = E(\pi \Pi(T))$. Since π is positive as a functional on L , if $\Pi(T) > 0$ then $\Pi(0) > 0$ and if $\Pi(T) = 0$ then $\Pi(0) = 0$.

On the other hand, let us suppose that there is no arbitrage. Let us consider the price-payoff vector space $V = \mathbb{R} \times L$. The (cost, pay-off) hyperplane is

$$M = \{(-\theta \cdot q, \theta \cdot P) : \theta \in \mathbb{R}^N\}.$$

The cone $K = \mathbb{R}_+ \times L_+$ contains all securities of non-positive price and non-negative payoff. If there is no arbitrage, then $K \cap M = \{0\}$.

By the separating hyperplane theorem, there exists a functional

$$F : V \rightarrow \mathbb{R}$$

such that $F(x) = 0$ for all $x \in M$ and $F(x) > 0$ for all $x \in K \setminus \{0\}$.

The Riesz representation of $F(x)$ is

$$F(v, c) = \alpha v + E(\phi \cdot c).$$

In terms of α and ϕ , we have that

$$-\alpha \theta \cdot q + \mathbb{E}(\phi \cdot (\theta \cdot P)) = 0$$

for all $\theta \in \mathbb{R}^N$. Hence

$$\pi \equiv \frac{\phi}{\alpha}$$

is a price deflator.