Interest Rate Theory.

Bonds, Yields

Consider a bond that, with a payment of $P(t,T)$ at time t, pays \$1 at time T (and has no other intermediate payments). If the interest rate r is assumed constant, then we would have

$$P(t,T) = e^{-r(T-t)}.$$

Hence,

$$r = -\frac{\log P(t,T)}{(T-t)}.$$

This is useful since it is P, not r that is observed (not quite: see next section). When r is not constant, we simply define the *yield rate*

$$r(t,T) = -\frac{\log P(t,T)}{(T-t)}.$$

Since r determines P, r determines the entire term structure.

As a function of T, r is smooth. As a function of t, it is a random.

Short rate

the cost of instantaneous borrowing:

$$r_t = r(t,t) = -\frac{\partial}{\partial T} \log P(t,T)\Big|_{T=t}.$$

- Similarity with stock prices
- loss of information.

Forward Rate

$f(t,T)$: "our prediction at t of r_T."
Consider the following futures contract:

- Agreement date: now (time t).
- Product to deliver: a zero-coupon bond B issued at T_1, paying \$1 at T_2.
- Delivery date: T_1.
- Price: $P(t, T_1, T_2)$ (Unknown).
- Payment date: T_1.

It has the only two cash flows:

- At time T_1: $P(t, T_1, T_2)$ (Unknown).
- At time T_2: \$1.

Consider a portfolio Π of 1 bond unit worth $P(t, T_2)$ each, and $-x$ bond units worth $P(t, T_1)$ each, with

$$x = \frac{P(t, T_2)}{P(t, T_1)}.$$

Because it costs nothing now, it has only two cash flows:

- At time T_1, an cash out-flow equal to $-x$ (or in-flow of x).
- At time T_2, a cash in-flow of \$1.

Hence,

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}.$$

The corresponding forward yield is, by definition

$$\begin{aligned} r(t, T_1, T_2) &= -\frac{\log P(t, T_1, T_2)}{T_2 - T_1} \\ &= -\frac{\log P(t, T_2) - \log P(t, T_1)}{T_2 - T_1}. \end{aligned}$$

The *forward rate* is defined to be

$$\begin{aligned} f(t,T) &= r(t,T,T) \\ &= -\frac{\partial}{\partial T} \log P(t,T). \end{aligned}$$

The term structure is reconstructed from f as follows:

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

Bootstrapping

In practice, bonds pay coupons. Hence, the calculation of the yield needs to be modified.

Example (J. Hull) Consider the following set of bonds, with \$100 principal and associated prices, with coupons paid every six months as described below:

Time to Maturity	annual coupon	price
3 mo.	\$0	\$97.5
6 mo.	\$0	\$94.9
1 yr.	\$0	\$90.0
1.5 yr.	\$8	\$96.0
2.0 yr.	\$12	\$101.6
2.75 yr.	\$10	\$99.8

Solving the obvious set of (non–linear equations), we can arrive at the yield curve with values given by

Term	annualized rate
3 mo.	10.12%
6 mo.	10.47%
1 yr.	10.54%
1.5 yr.	10.68%
2 yr.	10.81%
2.75 yr.	10.87%