Portfolio Credit Risk

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The Goodrich-Rabobank Swap 1983

- **U.S. Savings Banks**
  - 3.5 million (11% fixed)
  - Once a year
  - LIBOR + 0.5% (Semi)
  - B.F. Goodrich
  - BBB-rated

- **Belgian Dentists**
  - 5.5 million
  - Once a year
  - 11% annual
  - Rabobank
  - AAA-rated

**Swap**
- (LIBOR - x)% Semiannual
- Morgan Guarantee Trust

**Swap**
- (LIBOR - y)% Semiannual
Review of Basic Concepts

Credit Loss
Credit VaR
Credit Models
KMV and Merton Model
Exercises and Examples

Time Value of Money
Credit: Premium and Spread
A Two-State Markov Model
Cash Flow Valuation

Fundamental Principle:

**TIME IS MONEY**

The present value of cash flows is given by the value equation:

\[
\text{Value} = \sum_{i=1}^{n} p_i e^{-r_i t_i}
\]

Where:

- \( n \) is the number of payments
- \( p_i \) is the amount paid at time \( t_i \)
- \( r_i \) is the continuously compounded interest rate at time \( t_i \)

Equation (1) assumes payments will occur with probability 1 (no default risk)
The discounted value of cash flows, when there is probability of default, is given by:

\[
\text{Value} = \sum_{i=1}^{n} p_i e^{-r_i t_i} q_i
\]  

(2)

In the equation above, \( q_i \) denotes the probability that the counterparty is solvent at time \( t_i \). A large default risk (i.e. a small \( q \)) implies that:

1. For a fixed set of \( p_i \)'s the discounted present value will always be less than or equal to the value equation (equation (1))

2. To preserve the same present value of cashflows as in equation (1) the cashflows \( \{p_i\}_{i=1}^{n} \) need to be increased. The amount by which each payment is increased is \( q_i^{-1} \). This is the credit premium at time \( t_i \).
The credit spread.

- Since $q_i \leq 1$ we can write $q_i$ as:

$$q_i = e^{-h_i t_i}$$  (3)

which implies:

$$h_i = \frac{-\ln(q_i)}{t_i},$$

where $h_i$ is the credit spread at time $t_i$.

- The value of a loan with cashflows $\{p_i\}_{i=1}^n$ at times $\{t_i\}_{i=1}^n$ and credit spread $\{h_i\}_{i=1}^n$ is:

$$Value = \sum_{i=1}^n p_i e^{-(r_i+h_i)t_i}$$  (4)
Example: Default Yield Curve

A senior unsecured BB rated bond matures exactly in 5 years, and is paying an annual coupon of 6%

<table>
<thead>
<tr>
<th>Category</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>3.60</td>
<td>4.17</td>
<td>4.73</td>
<td>5.12</td>
</tr>
<tr>
<td>AA</td>
<td>3.65</td>
<td>4.22</td>
<td>4.78</td>
<td>5.17</td>
</tr>
<tr>
<td>A</td>
<td>3.72</td>
<td>4.32</td>
<td>4.93</td>
<td>5.32</td>
</tr>
<tr>
<td>BBB</td>
<td>4.10</td>
<td>4.67</td>
<td>5.25</td>
<td>5.63</td>
</tr>
<tr>
<td>BB</td>
<td>5.55</td>
<td>6.02</td>
<td>6.78</td>
<td>7.27</td>
</tr>
<tr>
<td>B</td>
<td>6.05</td>
<td>7.02</td>
<td>8.03</td>
<td>8.52</td>
</tr>
<tr>
<td>CCC</td>
<td>15.05</td>
<td>15.02</td>
<td>14.03</td>
<td>13.52</td>
</tr>
</tbody>
</table>

Table: One-year forward zero-curves for each credit rating (%)*

*Source: Creditmetrics, JP Morgan
Solution: Default Yield Curve

Using the on the previous slide find the 1-year forward price of the bond, if the obligor stays BB.

Solution (Default Yield Curve)

\[
V_{BB} = 6 + \frac{6}{1.0555} + \frac{6}{1.0602^2} + \frac{6}{1.0678^3} + \frac{106}{1.0727^4} = 102.0063
\]
First Model: Two Credit States

A simple two credit state model, some considerations and assumptions:

- What is the credit spread?
- Assume only 2 possible credit states: solvency and default.
- Assume the probability of solvency in a fixed period (one year, for example), conditional on solvency at the beginning of the period, is given by a fixed amount $q$. For period $t_{i+1}$ we have:

\[
\Pr(\text{Solvent at time } t_{i+1}|\text{Solvent at time } t_i) = q_i
\]

According to this model, we have:

\[
q_i = q^{t_i}
\]

which gives rise to a constant credit spread:

\[
h_i = h = -\ln(q)
\]
The General Markov Model

In other words, when the default process follows a Markov Chain the probabilities of default/solvency for period \((t_i, t_{i+1}]\) are given by the matrix:

<table>
<thead>
<tr>
<th></th>
<th>Solvency</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solvency</td>
<td>(q)</td>
<td>(1 - q)</td>
</tr>
<tr>
<td>Default</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table:** Markov Chain for the Constant Credit Spread \(h_i = h = -\ln q\)
Government Bonds
Sovereign default risk
Setup

Consider:
- Government bonds providing a spread $h$ over a risk-free bond
- Assume they are one-year zero coupon bonds
- Recovery rate $R$ of 50%
- (Risk Neutral) Probability of solvency after one year equal to $q$

Because of the recovery rate, we need to rewrite (2) as

$$V = N e^{-r} q + R \cdot N e^{-r} (1 - q) \quad (5)$$

Using the spread:

$$V = N e^{-(r+h)} \quad (6)$$

Therefore, solving for $V$:

$$q = \frac{e^{-h} - R}{1 - R} \quad (7)$$
Assuming a recovery rate of 50%, here are historical spread rates and the corresponding implied default probability according to this model:

<table>
<thead>
<tr>
<th>Date</th>
<th>Issuer</th>
<th>Spread</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/2012</td>
<td>Spain</td>
<td>419 bps</td>
<td>8.2%</td>
</tr>
<tr>
<td>5/2012</td>
<td>Portugal</td>
<td>909 bps</td>
<td>17%</td>
</tr>
<tr>
<td>5/2012</td>
<td>France</td>
<td>135 bps</td>
<td>2.7%</td>
</tr>
</tbody>
</table>

**Table:** Issuer risk
Transition Probabilities

Conditional probabilities, which give rise to a matrix with $n$ credit states:

$$
\begin{bmatrix}
    p_{11} & p_{12} & \cdots & \cdots & p_{1n} \\
    p_{21} & p_{22} & \cdots & \cdots & p_{2n} \\
    p_{31} & \cdots & \cdots & \cdots & \vdots \\
    \vdots & \cdots & \cdots & \cdots & \vdots \\
    p_{n1} & \cdots & \cdots & \cdots & p_{nn}
\end{bmatrix}
$$

$p_{ij}$ is the conditional probability of changing from state $i$ to state $j$. More formally:

$$p_{ij} = \Pr(\text{State } j|\text{State } i)$$
Credit rating agencies:

- There are corporations whose business is to rate the credit quality of corporations, governments and also specific debt issues.

- The main ones are:
  1. Moody’s Investors Service
  2. Standard & Poor’s
  3. Fitch IBCA
  4. Duff and Phelps Credit Rating Co
Credit Rating Agencies II

S & P’s Rating System

AAA - Highest Quality; Capacity to pay interest and repay principal is extremely strong

AA  - High Quality
   A  - Strong payment capacity

BBB - Adequate payment capacity

BB  - Likely to fulfill obligations; ongoing uncertainty

B   - High risk obligations

CCC - Current vulnerability to default

D   - In bankruptcy or default, or other marked shortcoming
**Standard and Poor’s Markov Model**

Multi-state transition matrix for Standard and Poor’s Markov Model:

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.9081</td>
<td>0.0833</td>
<td>0.0068</td>
<td>0.0006</td>
<td>0.0012</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>AA</td>
<td>0.0070</td>
<td>0.9065</td>
<td>0.0779</td>
<td>0.0064</td>
<td>0.0006</td>
<td>0.0014</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>A</td>
<td>0.0009</td>
<td>0.0227</td>
<td>0.9105</td>
<td>0.0552</td>
<td>0.0074</td>
<td>0.0026</td>
<td>0.0001</td>
<td>0.0006</td>
</tr>
<tr>
<td>BBB</td>
<td>0.0002</td>
<td>0.0033</td>
<td>0.0595</td>
<td>0.8693</td>
<td>0.0530</td>
<td>0.0117</td>
<td>0.0012</td>
<td>0.0018</td>
</tr>
<tr>
<td>BB</td>
<td>0.0003</td>
<td>0.0014</td>
<td>0.0067</td>
<td>0.0773</td>
<td>0.8053</td>
<td>0.0884</td>
<td>0.0100</td>
<td>0.0106</td>
</tr>
<tr>
<td>B</td>
<td>0.0000</td>
<td>0.0011</td>
<td>0.0024</td>
<td>0.0043</td>
<td>0.0648</td>
<td>0.8346</td>
<td>0.0407</td>
<td>0.0520</td>
</tr>
<tr>
<td>CCC</td>
<td>0.0022</td>
<td>0.0000</td>
<td>0.0022</td>
<td>0.0130</td>
<td>0.0238</td>
<td>0.1124</td>
<td>0.6486</td>
<td>0.1979</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Long Term Transition Probabilities

Transition probabilities:

- The transition probability between state \( i \) and state \( j \), in two time steps, is given by:

\[
p_{ij}^{(2)} = \sum_k p_{ik} \cdot p_{kj}
\]

- In other words, if we denote by \( A \) the one-step conditional probability matrix, the two-step transition probability matrix is given by:

\[
A^2
\]
Transition Probabilities in General

- If $A$ denotes the transition probability matrix at one step (on year, for example), the transition probability after $n$ steps (30 is especially meaningful for credit risk) is given by:

  $$A^n$$

- For the same reason, the quarterly transition probability matrix should be given by:

  $$A^{1/4}$$

- This gives rise to some important practical issues.
Matrix Expansion

We can expand a matrix as follows:

\[
A^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} (A - 1)^k
\]

Where

\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}
\]
Example: PRM Exam Question

Example: Transition Matrix Problem

The following is a simplified transition matrix for four states:

<table>
<thead>
<tr>
<th>Starting State</th>
<th>Ending State</th>
<th>Total Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>0.97</td>
<td>0.03</td>
</tr>
<tr>
<td>B</td>
<td>0.02</td>
<td>0.93</td>
</tr>
<tr>
<td>C</td>
<td>0.01</td>
<td>0.12</td>
</tr>
<tr>
<td>Default</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Calculate the cumulative probability of default in year 2 if the initial rating in year 0 is B.
Credit Loss
Credit Exposure

**Definition (Credit Exposure)**

*Credit exposure* is the maximum loss that a portfolio can experience at any time in the future, taken with a certain level of confidence.
Example: Credit Exposure

Evolution of the Mark-to-Market of a 20-Month Swap, where:
-99% Exposure
-95% Exposure
Recovery Rate:

- When default occurs, a portion of the value of the portfolio can usually be recovered.
- Because of this, a recovery rate is always considered when evaluating credit losses, more specifically:

**Definition (Recovery Rate)**

The *recovery rate* \( (R) \) represents the percentage value which we expect to recover, given default.
Similarly:

- We can define a related term called **loss given default**:

**Definition (Loss-Given Default)**

*Loss-given default* ($LGD$) is the percentage we expect to lose when default occurs. Mathematically this is equivalent to:

$$ R = 1 - LGD $$

- In both cases $R$ and $LGD$ may be modelled as random variables. However in simple exercises one may assume they are constant.
Recovery Rate for Bond Tranches

For corporate bonds, there are two primary studies of recovery rates which arrive at similar estimates (Carty & Lieberman and Altman & Kishore).

This study has the largest sample of defaulted bonds that we know of:

<table>
<thead>
<tr>
<th>Seniority Class</th>
<th>Carty and Liberman Study</th>
<th>Altman and Kishore Study</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number</td>
<td>Average</td>
</tr>
<tr>
<td>Senior Secured</td>
<td>115</td>
<td>$53.80</td>
</tr>
<tr>
<td>Senior Unsecured</td>
<td>278</td>
<td>$51.13</td>
</tr>
<tr>
<td>Senior subordinated</td>
<td>198</td>
<td>$38.52</td>
</tr>
<tr>
<td>Subordinated</td>
<td>226</td>
<td>$32.74</td>
</tr>
<tr>
<td>Junior Subordinated</td>
<td>9</td>
<td>$17.09</td>
</tr>
</tbody>
</table>

Table: Recovery Statistics by Seniority Class

Par (face value) is $100.00
Comments Regarding Studies

The table in the previous slide shows:

- The subordinated classes are appreciably different from one another in their recovery realizations.

- In contrast, the difference between secured versus unsecured debt is not statistically significant. It is likely that there is a self-selection affect here.

- There is a greater chance for security to be requested in the cases where an underlying firm has questionable hard assets from which to choose.
Let's develop the notion of default probability (frequency):

- Each counterparty has a certain probability of defaulting on their obligations.
- Some models include a random variable which indicates whether the counterparty is solvent or not.
- Other models use a random variable which measures the credit quality of the counterparty.
- For the moment, we will denote by \( \mathbb{I}\{\text{Counterparty Defaults?}\} \) the random variable which is 1 when the counterparty defaults, and 0 when it does not, i.e.:

\[
\mathbb{I}\{\text{Counterparty Defaults?}\} = \begin{cases} 
1 & \text{Counterparty Defaults} \\
0 & \text{Counterparty Does Not Default}
\end{cases}
\]
Default Probability (Frequency) Continued...

Continuing ...

- The modelling of how \( \mathbb{I}\{\text{Counterparty Defaults?}\} \) changes from 0 to 1 will be dealt with later.
Measuring the Distribution of Credit Losses I

For an instrument or portfolio with only one counterparty we define:

\[
\text{Credit Loss} = \mathbb{I}\{\text{Counterparty Defaults?}\} \times \text{Credit Exposure} \times LGD
\]

Note that:

- **Credit-Loss(Random Variable):** Depends on the credit quality of the counterparty
- **Credit Exposure(Number):** Depends on the market risk of the instrument or portfolio
- **LGD(number):** Usually this number is a universal constant (55%) but more refined models relate it to the market and the counterparty
For a portfolio with several counterparties we define:

\[
\text{Credit Loss} = \sum_i \mathbb{I}\{\text{Counterparty } i \text{ Defaults?}\} \times \text{Credit Exposure}_i \times LGD
\]

Note that:

- **Credit-Loss(Random Variable):** Normally different for different counterparties
- **Credit Exposure(Number):** Normally different for different portfolios, same for the same portfolios
- **LGD(number):** Usually this number is a universal constant (55%) but more refined models relate it to the market and the counterparty
Net Replacement Value:

- The traditional approach to measuring credit risk is to consider only the net replacement value (NRV):\[ NRV = \sum_{i} (\text{Credit Exposures})_i. \]

- This is a rough statistic, which measures the amount that would be lost if all counter-parties default at the same time, and at the time when all portfolios are worth most, and with no recovery rate.

Net Replacement Value: The traditional approach to measuring credit risk is to consider only the net replacement value (NRV)
Credit Loss Distribution

The credit loss distribution is often very complex:

- As with Markowitz theory, we try to summarize its statistics with two numbers: its expected value ($\mu$), and its standard deviation ($\sigma$).
- In this context, this gives us two values:
  1. The expected loss ($\mu$)
  2. The unexpected loss ($\sigma$)
Credit VaR/ Worst Credit Loss

Worst credit loss:

- Worst Credit Loss represents the credit loss which will not be exceeded with some level of confidence, *over a certain time horizon*.

- A 95%-WCL of $5M on a certain portfolio means that the probability of losing more than $5M in that particular portfolio is exactly 5%.

Credit-VaR:

- CVaR represents the credit loss which will not be exceeded *in excess of the expected credit* loss, with some level of confidence *over a certain time horizon*:

  - A daily CVaR of $5M on a certain portfolio, with 95% means that the probability of losing more than the expected loss plus $5M in one day in that particular portfolio is exactly 5%.
Economic Credit Capital

**Capital:**
- Capital is traditionally designed to absorb *unexpected* losses
- **Credit Var**, is therefore, the measure of capital.
  - It is usually calculated with a one-year time horizon
- Losses can come from either defaults or migrations

**Credit Reserves:**
- Credit reserves are set aside to absorb *expected* losses
- **Worst Credit Loss** measures the sum of the capital and the credit reserves
- Losses can come from either expected losses

Prof. Luis Seco
Portfolio Credit Risk
Netting I

What is netting:

- When two counterparties enter into multiple contracts, the cashflows over all the contracts can be, by agreement, merged into one cashflow.

- This practice, called *netting*, is equivalent to assuming that when a party defaults on one contract it defaults in all the contracts simultaneously.
Netting II

Properties of netting:

- Netting may affect the credit-risk premium of particular contracts.

- Assuming that the default probability of a party is independent from the size of exposures it accumulates with a particular counter-party, the expected loss over several contracts is always less or equal than the sum of the expected losses of each contract.

- The same result holds for the variance of the losses (i.e. the variance of losses in the cumulative portfolio of contracts is less or equal to the sum of the variances of the individual contracts).

- Equality is achieved when contracts are either identical or the underlying processes are independent.
Expected Credit Loss: General Framework

In the general framework, the expected credit loss (ECL) is given by:

$$ECL = E[I\{\text{Counterparty } i \text{ Defaults?}\} \times CE \times LGD]$$

$$= \int \int \int [I \times CE \times LGD \times f(I, CE, LGD)] dI \cdot dCE \cdot dLGD$$

Note that:
- $f(I, CE, LGD)$ is the joint probability density function of the:
  - Default status ($I$)
  - Credit Exposure (CE)
  - Loss Given Default (LGD)
- The ECL is the expectation using the jpdf of $I, CE$ and $LGD$
Expected Credit Loss: Special Case

- Because calculating the joint probability distribution of all relevant variables is hard, most often one assumes that their distributions are independent.
- In that case, the ECL formula simplifies to:

\[
ECL = \underbrace{E[I]}_{\text{Probability of Default}} \times \underbrace{E[CE]}_{\text{Expected Credit Exposure}} \times \underbrace{E[LDG]}_{\text{Expected Severity}}
\] (10)
Example 1

Example (Commercial Mortgage)

Consider a commercial mortgage, with a shopping mall as collateral. Assume the exposure of the deal is $100M, an expected probability of default of 20% (std of 10%), and an expected recovery of 50% (std of 10%).

Calculate the expected loss in two ways:

1. Assuming independence of recovery and default (call it x)
2. Assuming a −50% correlation between the default probability and the recovery rate (call it y).

What is your best guess as to the numbers x and y?

a) $10M, y = $10M.

b) $10M, y = $20M.

c) $10M, y = $5M.

d) $10M, y = $10.5M.
Solution 1

Solution (Commercial Mortgage)

What is your best guess as to the numbers \( x \) and \( y \)?

a) \( x = 10M, \ y = 10M \).

First notice that the answer cannot be a) or c) since \( x \) has to be smaller than \( y \).

Now we can take one of two approaches to find the solution. One is the tree-based approach whereas the other uses the covariance structure of the random variables.

Only the tree based approach is considered.
Note: Tree Based Model

Under the tree-based model we assume:

1. Two equally likely future credit states, given by default probabilities of 30% and 10%.
2. Two equally likely future recovery rates states, given by 60% and 40%.
Tree Based Model Continued...

Solution (Continued: Tree Based Model)

\[-0.5 = p^{++} - p^{+-} - p^{-+} + p^{--}\]
\[0.5 = p^{++} + p^{+-}\]
\[0.5 = p^{-+} + p^{--}\]
\[0.5 = p^{++} + p^{--}\]

Solving for $p^{++}$, $p^{+-}$, $p^{-+}$, $p^{--}$:

\[p^{++} = 0.125, \quad p^{+-} = 0.375, \quad p^{-+} = 0.125, \quad p^{--} = 0.375\]
Solution (Continued—Correlating Default and Exposure)

Given a −50% correlation between the recovery rates and credit states, along with the probabilities $p^{++}$, $p^{+-}$, $p^{-+}$, $p^{--}$, the expected loss (EL) is:

\[
EL = 100M \times \left( 0.375 \times 0.6 \times 0.3 + 0.375 \times 0.4 \times 0.1 \right.
\]
\[
+ \left. 0.125 \times 0.4 \times 0.3 + 0.125 \times 0.6 \times 0.1 \right)
\]
\[
= 100M \times (0.0825 + 0.0225)
\]
\[
= 10.5M
\]
Consider the swap between Goodrich and MGT. Assume a total exposure averaging $10M (50% std), a default rate averaging 10% (3% std), fixed recovery (50%).

Calculate the expected loss in two ways:

1. Assuming independence of exposure and default (call it $x$).
2. Assuming a $-50\%$ correlation between the default probability and the exposure (call it $y$).

What is your best guess as to the numbers $x$ and $y$?

a) $x = 500,000$ $y = 460,000$.
b) $x = 500,000$ $y = 1M$.
c) $x = 500,000$ $y = 500,000$. 
Solution 2

Solution (Goodrich-Rabobank)

What is your best guess as to the numbers $x$ and $y$?

a) $x = \$500,000$, $y = \$460,000$.

b) $x = \$500,000$, $y = \$1M$.

c) $x = \$500,000$, $y = \$500,000$.

d) $x = \$500,000$, $y = \$250,000$.

Notice that the answer cannot be c) or d) since $x$ has to be larger than $y$. 
Solution

**Correlating Default and Exposure Using the tree-based model we assume:**

1. Two equally likely future credit states, given by default probabilities of 13% and 7%.
2. Two equally likely exposures, given by $15M and $5M.

*With a 50% correlation between them, the expected loss (EL) is:*

\[
EL = 0.5 \times (0.125 \times 15M \times 0.13 + 0.125 \times 5M \times 0.7 \\
+ 0.375 \times 15M \times 0.07 + 0.375 \times 5M \times 0.13) \\
= 0.5 \times (0.24M + 0.40M + 0.24M) \\
= 0.5 \times 0.88M \\
= 0.44M \times 10^6 \\
= 460,000
\]
Example (23-2: FRM exam 1998, Question 39)

“Calculate the 1 yr expected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%.”
Example 23-2: FRM Exam 1998, Question 39

Example (23-2: FRM exam 1998, Question 39)

“Calculate the 1 yr expected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%.”

Solution (Example 23-2)

Note that the recovery rate is $1 - 0.6 = 40\%$, this implies:

\[ 0.06 \times $100M \times 0.6 = $3.6M \]
Example (Variant of Example 23-2)

“Calculate the 1 yr unexpected loss of a $100M portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default of each issuer is 6% and the recovery rate for each issuer in the event of default is 40%. Assume, also, that the correlation between the issuers is

1. 100% (i.e., they are all the same issuer)
2. 50% (they are in the same sector)
3. 0% (they are independent, perhaps because they are in different sectors)”
Variation of Example 23-2 II

Solution (Variation of Example 23-2)

1. The loss distribution is a random variable with two states: default (loss of $60M, after recovery), and no default (loss of 0). The expectation is $3.6M. The variance is

\[ 0.06 \times (60M - 3.6M)^2 + 0.94 \times (0 - 3.6M)^2 = 200(M)^2 \]

The unexpected loss is therefore

\[ \sqrt{200} = 14M \]
Solution (Variation of Example 23-2 Continued...)

2. The loss distribution is a sum of 10 random variable, each with two states: default (loss of $6M, after recovery), and no default (loss of 0). The expectation of each of them is $0.36M. The standard deviation of each is (as before) $1.4M. 

The standard deviation of their sum is

\[ \sqrt{10} \times 1.4M = 5M \]

Note: the number of defaults is given by a Poisson distribution. This will be of relevance later when we study the CreditRisk+ methodology.
3. The loss distribution is a sum of 10 random variables $X_i$, each with two states: default (loss of $6M, after recovery), and no default (loss of 0). The expectation of each of them is $0.36M. The variance of each is (as before) 2. The variance of their sum is:

$$\text{Var} \left( \sum_i X_i \right) = \sum_{i,j} E[X_i X_j] - \mu_i \mu_j$$

$$= \sum_i \sigma_i^2 + \sum_{i \neq j} \sigma_{i,j}$$

$$= \sum_i \sigma_i^2 + \sum_{i \neq j} 0.5 \times \sigma_i \sigma_j$$
Example (23-3: FRM exam 1999)

"Which loan is more risky? Assume that the obligors are rated the same, are from the same industry, and have more or less the same sized idiosyncratic risk: A loan of

a) $1M with 50% recovery rate.
b) $1M with no collateral.
c) $4M with a 40% recovery rate.
d) $4M with a 60% recovery rate."
Solution (Example 23-3)

The expected exposures times expected LGD are:

a) $500,000
b) $1M
c) $2.4M. Riskiest.
d) $1.6M
Example (23-4: FRM Exam 1999)

“Which of the following conditions results in a higher probability of default?

a) The maturity of the transaction is longer  
b) The counterparty is more creditworthy  
d) The price of the bond, or underlying security in the case of a derivative, is less volatile.  
d) Both 1 and 2.”
Solution 23-4: FRM Exam 1999

Solution (Example 23-4)

a) True

b) False, it should be “less”, nor “more”

c) The volatility affects (perhaps) the value of the portfolio, and hence exposure, but not the probability of default (*)
Expected and unexpected losses must take into account, not just a static picture of the exposure to one cash flow, but the variation over time of the exposures, default probabilities, and express all that in today’s currency.

This is done as follows: the PV ECL is given by:

$$PV(ECL) = \sum_t E[CL_t] \times PV_t$$  \hfill (11)
**Expected Loss: An Approximation**

We can re-write formula (11) as:

\[ PV(ECL) = \sum_t E[CL_t] \times PV_t \]

\[ = \sum_t p_t \times E[CE_t] \times (1 - f) \times PV_t \]

Note that each of these numbers changes with time.

\[ \approx \text{Ave}_t\{p_t\} \times \text{Ave}_t\{E[CE_t]\} \times (1 - f) \sum_t PV_t \]

Each term is replaced by an amount independent of time: *their average*

**Remark**

In the book, the term \((1 - f)\) is assumed to be independent of time. In some situations, such as commercial mortgages, this will underestimate the credit risk.
Portfolio Credit Risk

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April 1, 2014
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     - Expected Losses
   - Expected Loss
   - Unexpected Loss
   - Credit Reserve

3. **Credit VaR**
   - Examples
   - Problems
A Worked-Out Example
Consider a bond issued from a default-prone party, paying two $5 coupons after the end of the second and fourth years. We assume throughout the duration of the bond the interest rates are 0% (this assumption simplifies discounting).

The default-prone party has a yearly default probability of 7% and when it defaults no money can be recovered (recovery rate = 1−severity = 0).

We assume that the default-free party maintains a risk-capital to cover the standard deviation of losses that is adjusted annually and that it demands a certain return on this risk-capital.
Survival and Default Probabilities

where

- \( D \) = Default.
- \( ND \) = Not Default.

* Nodes are one year apart.
Expected loss calculation:

- There are **two, equivalent in this case, ways** to compute the expected loss.
- Since the value of the contract is always non-negative to the default-free party we do not need to discard any future events (as already explained this not a limitation, as every contract can be decomposed into contracts that have always non-negative or non-positive value).
- One way to compute the expected loss is to compute the expected cashflows.
- Recall that there are two such cashflows:
  1. $5 at \( t = 2 \),
  2. $5 at \( t = 4 \).

but we also need to factor in the probabilities of default within
Continuing...

- There are two cashflows of $5 each, and the expected cashflow is:

\[ EC = 5 \cdot p_{ND\in(0,2]} + 5 \cdot p_{ND\in(0,4]} = 8.065 \]

where \( p_{ND\in(i,j]} \) is the probability that the default-prone party does not default in the time interval between years \( i \) and \( j \) \( (i < j) \).

- The expected loss is:

\[ EL = 10 - EC = 1.935 \]
An Equivalent Way to Calculate Expected Loss

The second way is to calculate loss:

- Is based on the yearly exposure:

  \[
  \text{Exposure(year } 1^-) = $10 \\
  \text{Exposure(year } 2^-) = $10 \\
  \text{Exposure(year } 3^-) = $5 \\
  \text{Exposure(year } 4^-) = $5 
  \]

  where no correction is due to discounting was included, since interest rates are flat at 0% and Exposure(year 1^-), the value of the contract just before year 1.
The expected losses are:

\[
EL = \text{Exposure(year 1)} \cdot p_{D \in (0,1]} + \text{Exposure(year 1)} \cdot p_{D \in (1,2]} \\
+ \text{Exposure(year 3)} \cdot p_{D \in (2,3]} + \text{Exposure(year 4)} \cdot p_{D \in (3,4]}
\]

\[
= 10 \times 0.07 + 10 \times 0.0651 + 5 \times 0.0605 + 5 \times 0.0563
\]

\[
= $1.935
\]

where \( p_{D \in (2,3]} \) is the probability that the default-prone party defaults in the time interval between years 2 and 3.
The Unexpected Loss

Recall that the **unexpected loss** is the **variance of the losses**, so:

\[
\mathbb{V}(L_{[0,1]}) = \text{Exposure(year 1)}^2 \cdot p_{D \in (0,1]} - \left( \text{Exposure(year 1)} \cdot p_{D \in (0,1]} \right)^2 \\
= (EL(1))^2 \left( \frac{1}{p_{D \in [0,1)}} - 1 \right) = 6.51
\]

\[
\mathbb{V}(L_{[1,2]}) = \text{Exposure(year 2)}^2 \cdot p_{D \in (1,2]} - \left( \text{Exposure(year 2)} \cdot p_{D \in (1,2]} \right)^2 \\
= (EL(2))^2 \left( \frac{1}{p_{D \in [1,2)}} - 1 \right) = 6.08
\]

\[
\mathbb{V}(L_{[2,3]}) = (EL(3))^2 \left( \frac{1}{p_{D \in [2,3)}} - 1 \right) = 1.42
\]

\[
\mathbb{V}(L_{[3,4]}) = (EL(4))^2 \left( \frac{1}{p_{D \in [3,4)}} - 1 \right) = 1.33
\]

where

\[
\mathbb{V}(X) = \text{Var}(X) = E \left[ X^2 \right] - \left( E[X] \right)^2
\]
Credit Reserve:

- If, for any example, a risk-capital of two standard deviations is required, the default-free party anticipates to use risk-capital equal to:
  1. $5.10 at year 0,
  2. $4.93 at year 1,
  3. $2.38 at year 2 and
  4. $2.31 at year 3.

- A yearly return of 10% on such capital leads to an additional surcharge of $1.47.

Remark

Notice that a high enough return rate would lead to the possibility of arbitrage (in this case arbitrage corresponds to an intial credit-risk premium of more than $10).
Credit VaR
Credit VaR:

- **Credit VaR** is the unexpected credit loss, at some confidence level, over a certain time horizon.

- If we denote by \( f(x) \) the distribution of credit losses over the prescribed time horizon (typically one year), and we denote by \( c \) the confidence level (i.e. 95%), then the Worst-Credit-Loss (WCL) is defined to be:

\[
\int_{WCL}^{\infty} f(x)dx = 1 - c
\]

and

Credit VaR = (Worst-Credit Loss) – (Expected Credit Loss)

 Leads to Reserve Capital
A risk analyst is trying to estimate the Credit VaR for a risky bond. The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon. Assume that the bond is valued at $1M one month forward, and the one year cumulative default probability is 2% for this bond.

What is your estimate of the Credit VaR for this bond assuming no recovery?

- a) $20,000
- b) $1,682
- c) $998,318
- d) $0
What is your estimate of the Credit VaR for this bond assuming no recovery?

a) $20,000  

b) $1,682  

c) $998,318  

d) $0

Why?

If $d$ is the monthly probability of default then:

1. $(1 - d)^{12} = 0.98$, so $d = 0.00168$,
2. $ECL = $1,682,
3. $WCL(0.999) = WCL(1 - 0.00168) = $1,000,000,
4. $CVaR = $1,000,000 - 1,682 = $998,318.$
A risk analyst is trying to estimate the Credit VaR for a portfolio of two risky bonds.

The Credit VaR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one month horizon.

Assume that both bonds are valued at $500,000 one month forward, and the one year cumulative default probability is 2% for each of these bonds.

What is your best estimate of the Credit VaR for this portfolio assuming no default correlation and no recovery?

- a) $841
- b) $1,682
- c) $10,000
- d) $249,159
What is your best estimate of the Credit VaR for this portfolio assuming no default correlation and no recovery?

a) $841  
b) $1,682  
c) $10,000  
d) $249,159

Why?

If $d$ is the monthly probability of default then:

$(1 - d)12 = (0.98)$, so $d = 0.00168$, 

$ECL = $839.70$, 

$WCL(0.999) = WCL(1-0.00168) = $250,000$, 

$\therefore CVaR = $250,000 - $840 = $249,159$.  

Credit Loss Distribution

As before, the monthly discount is $d = 0.00168$

The 99.9% loss quantile is about $500,000$

Also we have that:

- $EL = 839.70$
- $WCL(0.999) = WCL(1-0.00168) = 250,000$

$CVaR = 250,000 - 840 = 249,159$

<table>
<thead>
<tr>
<th>Default</th>
<th>Probability</th>
<th>Loss</th>
<th>$P \times L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Bonds</td>
<td>$d^2 = 0.00000282$</td>
<td>$500,000$</td>
<td>$1.4$</td>
</tr>
<tr>
<td>1 Bond</td>
<td>$2 \times d \times (1 - d) = 0.00336$</td>
<td>$250,000$</td>
<td>$830.70$</td>
</tr>
</tbody>
</table>
Example

- Consider a stock $S$ valued at $1$ today, which after one period can be worth $S_T$: $2$ or $0.50$.
- Consider also a convertible bond $B$, which after one period will be worth $\max(1, S_T)$.
- Assume the stock can default ($p = 0.05$), after which event $S_T = 0$ (no recovery).
- Determine which of the following three portfolios has the lowest 95%-Credit-VaR:
  1. $B$
  2. $B - S$
  3. $B + S$
Portfolio Credit Risk

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   - Examples
   - Probabilities

---

**Prof. Luis Seco**  
**Portfolio Credit Risk**
Credit Models
Portfolio: Credit Risk Models

- CreditMetrics
  - PMorgan
- CreditRisk+
  - Credit Suisse
- KMV
  - KMV
- CreditPortfolioView
  - McKinsey
Defining Characteristics

Risk Definition
- Default-mode models only take into account default as a credit event.
  
  (CR+,KMV)
- MtM models consider changes in market values and credit ratings as they affect the value of the instruments (CM,CPV)

Bottom-Up Vs. Top-Down
- **Top-down** models ignore details of each individual transaction and focus on the impact of each instrument on a large list of risk sources.
  → Appropriate for retail portfolios. (CPV)
- **Bottom-up** models focus on the risk profile of each instrument.
  → Appropriate for corporate or sovereign portfolios. (CM,CR+KMV)
CreditMetrics:

- Credit risk is driven by movements in bond ratings.
- Analyses the effect of movements in risk factors to the exposure of each instrument in the portfolio (instrument exposure sensitivity). Credit events are rating downgrades, obtained through a matrix of migration probabilities.
- Each instrument is valued using the credit spread for each rating class.
- Recovery rates are obtained from historical similarities.
- Correlations between defaults are inferred from equity prices, assigning each obligor to a combination of 152 indices (factor decomposition).
- All this information is used to simulate future credit losses.
- It does not integrate market and credit risk.
Simulation of one Asset: a Bond

<table>
<thead>
<tr>
<th>Bond Rating</th>
<th>Probability(%)</th>
<th>Value</th>
<th>$\sum_i p_i V_i$</th>
<th>$\sum_i p_i (V_i - m)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02</td>
<td>$109.37$</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>AA</td>
<td>0.33</td>
<td>$109.19$</td>
<td>0.36</td>
<td>0.01</td>
</tr>
<tr>
<td>A</td>
<td>5.95</td>
<td>$108.66$</td>
<td>6.47</td>
<td>0.15</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93</td>
<td>$107.55$</td>
<td>93.49</td>
<td>0.19</td>
</tr>
<tr>
<td>BB</td>
<td>5.30</td>
<td>$102.02$</td>
<td>5.41</td>
<td>1.36</td>
</tr>
<tr>
<td>B</td>
<td>1.17</td>
<td>$98.10$</td>
<td>1.15</td>
<td>0.95</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12</td>
<td>$83.64$</td>
<td>0.10</td>
<td>0.66</td>
</tr>
<tr>
<td>Default</td>
<td>0.18</td>
<td>$51.13$</td>
<td>0.09</td>
<td>5.64</td>
</tr>
</tbody>
</table>

Table: Figure 23-3 Building the Distribution of Bond Values*

Note:
- The boldface values are the statistics for the MtM.
- We also have that:

<table>
<thead>
<tr>
<th>99% CVaR</th>
<th>[$11, $24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2σ</td>
<td>$6</td>
</tr>
</tbody>
</table>

* Source: CreditMetrics
Correlations in CreditMetrics

Two counter-parties:

- Models were generated using 152 country indices, 28 country indices, 19 worldwide indices including:
  1. US Chemical Industries
  2. German Insurance Index
  3. German Banking Index

- Linear Regression

\[
\begin{align*}
  r_1 &= 0.9 \times r_{US,Ch} + k_1 \xi_1 \\
  r_2 &= 0.74 \times r_{GE,In} + 0.1 \times r_{GE,Ba} + k_2 \xi_2
\end{align*}
\]

- In the linear regression models above we further assume that the residuals \( \xi_1 \) and \( \xi_2 \) are uncorrelated.
The correlation on returns are however modelled as:

\[ \rho_{\text{def}}(r_1, r_2) = 0.90 \times 0.74 \rho(r_{US,Ch} r_{GE,ln}) + 0.90 \times 0.15 \rho(r_{US,Ch} r_{GE,Ba}) = 0.11 \]

Note:
Correlations on returns drive the correlations on credit ratings
Simulation of more than one Asset

Simulating more than one asset:

- Consider a portfolio consisting of $m$ counterparties, and a total of $n$ possible credit states.
- We need to simulate a total of $nm$ states; their multivariate distribution is given by their marginal distributions (as before) and the correlations given by the regression model.
- To obtain accurate results, since many of these states have low probabilities, large simulations are often needed.
- It does not integrate market and credit risk: losses are assumed to be due to credit events alone: for example,
  - Swaps exposures are taken to be their expected exposures.
  - Bonds are valued using today’s forward curve and current credit spreads for generated future credit ratings.
Exercise

Pricing the Goodrich swap using the CreditMetrics framework
The Full Swap

Setup:

- If we consider the full swap, we need to consider the default process $b$ and the interest rate process $r$.
- The random variable that describes losses is given by
  \[
  \text{Loss} = 50 \sum_{8 \text{ Years}} (11 - \text{libor}_t) e^{-r_t t} b_t.
  \]
- If we assume the credit process and the market process are independent, we get:
  \[
  \text{ECL} = 50 \sum_{8 \text{ Years}} \mathbb{E}(11 - \text{libor}_t) e^{-r_t t} \mathbb{E}[b_t].
  \]
- This will overestimate the risk in the case that the default process and the market process are negatively correlated.
Monte Carlo Approach:

- Correlation on market variables drive correlations of default events:
  \[ \rho(\text{libor}, \text{GR}) = -0.47 \]

- Then,
  \[ \rho(\text{Libor}_t, b_t) = -0.47 \]

and

\[ ECL = 50 \sum_{8 \text{ Years}} \left[ \mathbb{E}(11 - \text{libor}_t)_+ \right] e^{-r_t t} \mathbb{E}[b_t]. \]

is calculated with Monte-Carlo techniques.
CreditMetrics Approach:

- Assume a 1 year time horizon, and that we wish to calculate the loss statistics for that time horizon.
- Assume credit ratings with transition probabilities from BBB are given by:

<table>
<thead>
<tr>
<th>Bond Rating</th>
<th>Probability(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02</td>
</tr>
<tr>
<td>AAA</td>
<td>0.33</td>
</tr>
<tr>
<td>AAA</td>
<td>5.95</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93</td>
</tr>
<tr>
<td>BB</td>
<td>5.30</td>
</tr>
<tr>
<td>B</td>
<td>1.17</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12</td>
</tr>
<tr>
<td>Default</td>
<td>0.18</td>
</tr>
</tbody>
</table>

- Spreads are given by:

<table>
<thead>
<tr>
<th>Bond Rating</th>
<th>Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>25</td>
</tr>
<tr>
<td>AA</td>
<td>40</td>
</tr>
<tr>
<td>A</td>
<td>100</td>
</tr>
<tr>
<td>BBB</td>
<td>180</td>
</tr>
<tr>
<td>BB</td>
<td>250</td>
</tr>
<tr>
<td>B</td>
<td>320</td>
</tr>
<tr>
<td>CCC</td>
<td>500</td>
</tr>
<tr>
<td>Default</td>
<td></td>
</tr>
</tbody>
</table>
The loss statistics are summarized as follows:

<table>
<thead>
<tr>
<th>Credit Event</th>
<th>MtM Change in $K</th>
<th>Spread (bpi)</th>
<th>Pr(default) (%)</th>
<th>Pr(default) × MtM Change($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>155</td>
<td>25</td>
<td>0.02</td>
<td>31</td>
</tr>
<tr>
<td>AA</td>
<td>140</td>
<td>40</td>
<td>0.33</td>
<td>462</td>
</tr>
<tr>
<td>A</td>
<td>80</td>
<td>100</td>
<td>2.95</td>
<td>2360</td>
</tr>
<tr>
<td>BBB</td>
<td>0</td>
<td>180</td>
<td>86.93</td>
<td>0</td>
</tr>
<tr>
<td>BB</td>
<td>-70</td>
<td>250</td>
<td>5.3</td>
<td>-3710</td>
</tr>
<tr>
<td>B</td>
<td>-140</td>
<td>320</td>
<td>1.17</td>
<td>-1638</td>
</tr>
<tr>
<td>CCC</td>
<td>-320</td>
<td>500</td>
<td>0.12</td>
<td>-384</td>
</tr>
<tr>
<td>Default</td>
<td>-10000</td>
<td></td>
<td>0.18</td>
<td>-18000</td>
</tr>
</tbody>
</table>
Loss Statistics Over the Life of the Asset

- Expected exposures, and exposure quantiles (in the case of this swap) will generally decrease over the life of the asset.
  → They are pure market variables, which can be calculated with Monte Carlo methods.

- Probability of default, and the probability of other credit downgrades, increase over the life of the asset.
  → They are calculated, either with transition probability matrices, or with default probability estimations (Merton’s model, for instance).

- Discount factors will also decrease with time, and are given by the discount curve.

\[ PV - ECL = \sum_{t} E[CL_t] \times PV_t \]
Assume the $ECL=50,000$ and $UCL=200,000$ then,

<table>
<thead>
<tr>
<th>GR Swap</th>
<th>bps</th>
<th>$K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Captial at Risk (UL, or CVaR)</td>
<td></td>
<td>200</td>
</tr>
<tr>
<td>Cost of Capital is (15%-8%=7%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Required Net Income (8 Years)</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>Tax (40%)</td>
<td></td>
<td>75</td>
</tr>
<tr>
<td>Pretax Net Income</td>
<td></td>
<td>175</td>
</tr>
<tr>
<td>Opearating Costs</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>Credit Provision (ECL)</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Hedging Costs</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Required Revenue</td>
<td>0.5</td>
<td>327</td>
</tr>
</tbody>
</table>
CreditRisk+:

- Uses only two states of the world: default/no-default.
- But allows the default probability to vary with time.
  As we saw in the review, if one considers only default/no-default states, the default probability must change with time to allow the credit spread to vary (otherwise, the spread is constant, and does not fit observed spreads).
- Defaults are Poisson draws with the specified varying default probabilities.
- Allows for correlations using a sector approach, much like CreditMetrics.
  - However, it divides counter-parties into homogeneous sectors within which obligors share the same systematic risk factors.
- Severity is modeled as a function of the asset; assets are divided into severity bands.
CreditRisk+ Continued...

- It is an analytic approach, providing quick solutions for the distribution of credit losses.
- No uncertainty over market exposures.
CreditRisk+: Introductory Considerations

If we have a number of counter-parties A, each with a probability of default given by a fixed $P_A$, which can all be different, then the individual probability generating function is given by:

$$F_A(z) = \sum_{n} [\text{Prob } n \text{ Defaults}] z^n$$

$$= 1 - p_A + p_A z$$
If defaults are independent of each other, the generating function of all counter-parties is:

\[ F(z) = \sum_{n} [Pr(n \text{ Defaults})] z^n \]

\[ = \prod_{A} F_{Z}(z) = \prod_{A} (1 - p_A(1 - z)) \]

\[ \approx \exp \left( \sum_{A} p_A(1 - z) \right) \]

\[ = \exp(1 - \mu z) \]

\[ = \sum_{n} \frac{e^{-\mu} \mu^n}{n!} z^n \]

Note that: the standard deviation of the Poisson distribution is given by \( \sqrt{\mu} \)
Statistics from 1970 to 1996:

<table>
<thead>
<tr>
<th>Rating</th>
<th>Average Default Probability (%)</th>
<th>Standard Deviation(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Aa</td>
<td>0.03</td>
<td>0.01</td>
</tr>
<tr>
<td>A</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>Baa</td>
<td>0.12</td>
<td>0.3</td>
</tr>
<tr>
<td>Ba</td>
<td>1.36</td>
<td>1.3</td>
</tr>
<tr>
<td>B</td>
<td>7.27</td>
<td>5</td>
</tr>
</tbody>
</table>
CreditRisk+: Implementation

Each obligor is attached to an economic sector:

- The average default rate of sector \( k \) is given by a number \( x_k \), which is assumed to follow a Gamma distribution with parameters \( \alpha_k \) and \( \beta_k \).
- This yields a probability of default for each obligor in a sector which has a mean \( \mu_k \) and standard deviation \( \sigma_k \):
  - For each sector \( k \):
    \[
    \alpha = \frac{\mu^2}{\sigma^2}, \quad \beta = \frac{\sigma^2}{\mu}
    \]
    with density function:
    \[
    f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} e^{-x/\beta} x^{\alpha-1}
    \]
    and generating function given by:
    \[
    F(z) = \int_0^\infty e^{x(z-1)} f(x) dx = \left( \frac{1 - p}{1 - pz} \right)^\alpha, \quad p = \frac{\beta}{1 + \beta}
    \]
According to this, the probability of $n$ defaults is given by the $n^{th}$ term in the power series expansion of the generating function:

$$F(z) = \sum_{n=1}^{\infty} \binom{n + \alpha - 1}{n} p^n (1 - p)^{\alpha} z^n$$

$$\Pr(n \text{ defaults})$$
For the entire portfolio, with exposures to several market indices \( k \), the generating function is given by:

\[
F(z) = \prod F_k(z) = \prod \left( \frac{1 - p_k}{1 - p_k z} \right)^{\alpha_k}
\]
CreditRisk+: Loss Distribution

- It goes from the distribution of default events to the loss distribution, by introducing a unit loss concept, with their associated distributions and generating functions, as follows:
  - It breaks up the exposure of the portfolio into \( m \) exposure bands, each band with an exposure of \( v \) units.
  - The individual bands are assumed to be independent, and have a generating function equal to:

**Generating Function: Individual Bands**

\[
G_j(z) = \sum_n [\text{Prob } n \times v_j \text{ units loss}] z^{nv_j} \\
= \sum_n e^{-\mu_j} \frac{\mu_j^n}{n!} z^{nv_j} \\
= \exp(-\mu_j + \mu_j z^{v_j}) = \exp(F_{\text{Poisson}}(P_j(z)))
\]
For the entire portfolio, with exposures to several market indices $k$, and all exposure bands represented by levels $j$, we have a final explicit expression given by:

**Generating Function: Portfolio**

$$G(z) = \prod_{k} F_{k} \left[ \sum_{j} P_{k,j}(z) \right]$$
KMV and Merton Model
The Merton Model

Merton Model:

- Merton (1974) introduced the view that equity value is a call option on the value of the assets of the firm, with a strike price equal to the firm's debt.

- In particular, the stock price embodies a forecast of the firm's default probabilities, in the same way that an option embodies an implied forecast of the option being exercised.
A Simple Setting

- If the value of the firm is less than $K$:
  1. The bondholders get the value of the firm $V$ and
  2. Equity value is 0.
  → The firm would then be in default.
Equity Values and Option Prices

The firm value can be determined as follows:

- In our simple example before, stock value at expiration is:

\[ S_T = \max(V_T - K, 0) \]

- Since the firm’s value equals equity plus bonds, we have that the value of the bond is:

\[ B_T = V_t - \max(V_t - K, 0) \]
\[ = \min(V_T, k) \]
Similarly, the bond value can be expressed as:

$$B_T = K - \max(K - V_T, 0)$$

- In other words, a long position in a risky bond is equivalent to a long position in a risk-free bond plus a short put option.
- The short put option is really a credit derivative, same as the risky bond.
- This shows that corporate debt has a payoff similar to a short option position, which explains the left skewness in credit losses.
- It also shows that equity is equivalent to an option on the values assets; due to the limited liability of the firm, investors can lose no more than their original investment.
Pricing Equity

Pricing equity, assumptions:

- We assume the firm’s value follows a geometric Brownian motion process:
  \[ dV = \mu V dt + \sigma V dz \]

- If we assume no transaction costs (including bankruptcy costs)
  \[ V = B + S \]

- Since stock price is the value of the option on the firm’s assets, we can price it with the Black-Scholes methodology, obtaining:

\[
S = VN(d_1) - Ke^{-rt}N(d_2)
\]

(12)

where

\[
d_1 = \frac{-\ln(Ke^{-e\tau}/V) + \sigma \sqrt{\tau}}{\sigma \sqrt{\tau}} + d_2 = d_1 - \frac{\sigma \sqrt{\tau}}{2},
\]
Pricing equity continued...

Note that:

$$S = VN(d_1) - Ke^{-rt}N(d_2)$$

where

$$d_1 = \frac{-\ln\left(\frac{Ke^{-e\tau}}{V}\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

- $\sigma\sqrt{\tau}$ is the asset volatility
- $\frac{Ke^{-r\tau}}{V}$ is the Leverage: Debt/Value Ratio
Asset Volatility:

- In practice, only equity volatility is observed, not asset volatility, which we must derive as follows:

- The hedge ratio

\[ dS = \frac{\partial S}{\partial V} dV \]

yields a relationship between the stochastic differential equations for \( S \) and \( V \), from where we get

\[ \sigma_S S = \sigma_V V \frac{\partial S}{\partial V} \]

and

\[ \sigma_V = \sigma_S \frac{S \partial V}{V \partial S} \]
Pricing Debt

Pricing debt:

- The value of the bond is given by:

\[ B = V - S \]

or

\[ B = Ke^{-rt} N(d_2) + N[1 - N(d_1)] \]

- This is the same as:

\[ \frac{B}{Ke^{-R\tau}} = N(d_2) + \left(\frac{V}{Ke^{-r\tau}}\right) N(-d_1) \]

- Note that:
  - Probability of exercising the call, or probability that the bond will not default
Credit Loss:

- The expected credit loss is the value of the risk-free bond minus the risky bond:

\[
E_{CL} = Ke^{-r\tau} - Ke^{-r\tau}N(d_2) - V[1 - N(d_1)] \\
= Ke^{-r\tau}N(-d_2) - VN(-d_2)
\]

- This is the same as:

\[
E_{CL} = N(-d_2) \left[ Ke^{-r\tau} - V \frac{N(-d_1)}{N(-d_2)} \right] \\
= \text{Probability } \times \text{ Exposure } \times \text{ Loss-Given-Default}
\]

Note that:

- Probability of not exercising the call, or probability that the bond will default.
- \( PV \) of face value of the bond.
Advantages:

- Relies on equity prices, not bond prices: more companies have stock prices than bond issues.
- Correlations among equity prices can generate correlations among default probabilities, which would be otherwise impossible to measure.
- It generates movements in EDP that can lead to credit ratings.
Disadvantages:

- Cannot be used for counterparties without traded stock (governments, for example).
- Relies on a static model for the firms capital and risk structure:
  - The debt level is assumed to be constant over the time horizon.
  - The extension to the case where debt matures are different points in time is not obvious.
- The firm could take on operations that will increase stock price but also its volatility, which may lead to increased credit spread; this is in apparent contradiction with its basic premise, which is that higher equity prices should be reflected in lower credit spreads.
KMV:

- KMV was a firm founded by Kealhofer, McQuown and Vasicek, (sold recently to Moodys), which was a vendor of default frequencies for 29,000 companies in 40 different countries.
- Much of what they do is unknown.
- Their method is based on Merton’s model: the value of equity is viewed as a call option on the value of the firm’s assets.
- Basic model inputs are:
  1. Value of the liabilities (calculated as liabilities (t-1 year) plus one half of long term debt)
  2. Stock value
  3. Volatility
  4. Assets
Basic Terms

Market Value Assets

Possible Asset Value Path

Distribution of Asset Value at the Horizon

Default Point

Time

Prof. Luis Seco
Portfolio Credit Risk
Basic Formula: Distance to Default

\[
\text{Distance to Default} = \frac{\text{Market Value of Assets} - \text{Default Point}}{\text{Market Value of Assets} \times \text{Asset Volatility}}
\]
The exact value of the distance to default is calculated as:

$$ \text{Distance to Default}_t = \frac{\ln \left( \frac{V_A}{L_t} \right) - \frac{\sigma_A^2}{2} t}{\sigma_A \sqrt{T}}, $$

where:

- $V_A$: Market value of assets.
- $L_t$: Market value of liabilities maturing at time $t$.

It is often difficult to the exact distance to default, so we can use the following approximation:

$$ \text{Distance to Default}_t \approx \frac{\ln \left( \frac{V_A}{L_t} \right)}{\sigma_A} \approx \frac{V_A}{L_t} - 1 $$
Example

Consider a firm with:

- $100M Assets,
- $80M liabilities,
- Volatility of $10M (annualized)

Then the distance from default is calculated as:

\[
\frac{A - K}{\sigma} = 2
\]

The default probability is then 0.023 (using a Gaussian).
Example (Continued...)

The capital requirement under BIS is:

\[
K = \text{LGD} \times \left[ N \left( \frac{N^{-1}(PD) - \sqrt{R} N^{-1}(0.999)}{\sqrt{1 - R}} \right) - PD \right] \times MF
\]

where:

- \( N \): Cumulative Normal,
- \( \sqrt{R} \): 1-Factor asset correlation,
- \( MF \): Maturity function:
  - Empirically adjusts for the maturity of the portfolio.
Exercise 1: The Merton Model

Example (The Merton Model)

- Consider a firm with total asset worth $100, and asset volatility equal to 20%.
- The risk free rate is 10% with continuous compounding.
- Time horizon is 1 year.
- Leverage is 90% (i.e., debt-to-equity ratio 900%)

Find:

1. The value of the credit spread.
2. The risk neutral probability of default.
3. Calculate the PV of the expected loss.
Solution Part 1: Finding the Credit Spread

Solution (The Merton Model-Finding the Credit Spread)

- A leverage of 0.9 implies that
  \[ Ke^{-0.1}/V = 0.9 \]
  which says that \( K = 99.46 \).

- Using Black-Scholes, we get that the call option is worth \( S = 13.59 \).

- The bond price is then
  \[ B = V - S = 100 - 13.59 = 86.41 \]
  for a yield of
  \[ \ln \left( \frac{K}{B} \right) = \ln \left( \frac{99.46}{86.41} \right) = 14.07\% \]
  or a credit spread of 4.07\%.
Solution Part 2: Option Calculation

Solution (The Merton Model-Option Calculation)

- Underlying Data:
  - Equity
  - Stock Price: 100.00
  - Volatility (% per year): 20.00%
  - Risk-Free Rate (% per year): 10.00%

- Option Data:
  - Option Type: Analytic European
  - Time to Exercise: 1.0000
  - Exercise Price: 99.46

- Graph Results:
  - Vertical Axis:
    - Theta
  - Horizontal Axis:
    - Asset price
    - Minimum X value: 80
    - Maximum X value: 120

- Table of Option Prices and Gains:
  - Price: 13.5923574
  - Delta (per $): 0.73459442
  - Gamma (per $ per $): 0.01638677
  - Vega (per %): 0.32773533
  - Theta (per day): -0.0253837
  - Rho (per %): 0.59077085

Graph showing the relationship between asset price and option price/theta.
Solution Part 2: The Risk Neutral Probability of Default

Solution (The Merton Model-Risk Neutral Probability of Default)

The risk neutral probability of default is given by:

\[ N(d_2) = 0.6653, \quad EDF = 1 - N(d_2) = 33.47\% \]
Solution Part 3: The Expected Loss

Solution (The Merton Model-The Expected Loss)

The expected loss is given by:

\[ ECL = N(-d_2) \left[ Ke^{-r\tau} - V \frac{N(-d_1)}{N(-d_2)} \right] \]

\[ = 0.3347 \times \left[ $90 - $100 \times \frac{0.2653}{0.3347} \right] \]

\[ = $3.96 \]
Additional Considerations

Variations of the same problem:

- If debt-to-equity ratio is 233%, the spread is 0.36%.
- If debt-to-equity ratio is 100%, the spread is about 0.
- In other words, the model fails to reproduce realistic, observed credit spreads.
Example (The Goodrich Corporation)

The following information about the Goodrich corporation is available to us:

- From the Company’s financials:
  - Debt/equity ratio: 2.27
  - Shares out: 117,540,000.
  - Expected dividend: $0.20/share.

- From NYSE, ticker symbol GR:
  - Stock volatility: 49.59%
  - Real rate of return (3 years): 0.06%
  - Share price: $17.76 (May 2003)

- From Interest rate market:
  - Annual risk free rate: 3.17%
Example (The Goodrich Corporation Continued...)

- Stock and Company Value:
  \[ S = 117,540,000 \times 17.76 \]
  \[ = 2,087B \]
  \[ V = S + B = 3.27S = 6.826B \]

- Current Debt: $4.759B

- Future debt (strike price):
  \[ K = 4.759e^{0.0317} = 4.912 \]

- Dividend:
  \[ \text{Dividend} = 0.20 \times 117,540,000 \]
  \[ = 23,508,000 \]
Solution (Boostrapping Asset Volatility)

\[
\sigma_V = \sigma_S \frac{S \partial V}{V \partial S}
\]
Booststrapping Asset Volatility (Iterative Process)

Solution (Booststrapping Asset Volatility -Iterative Process)

Prof. Luis Seco  Portfolio Credit Risk
McKinsey’s Credit Portfolio View

- Introduced in 1997.
- Considers only default/no-default states, but probabilities are time dependent, given by a number \( p_t \).
- It is calculated as follows: given macroeconomic variables \( x_k \), it uses a multifactor model (Wilson 1997)

\[
y_t = \alpha + \sigma_k \beta_k x_k
\]

to assign a debtor a country, industry and rating segment.
- It assigns a probability of default given by

\[
p_t = \frac{1}{1 + \exp(y_t)}
\]

- The model is convenient to model default probabilities in macroeconomic contexts, but it is inefficient for corporate
Suppose:

- We have three models:
  - CM-CreditMetrics
  - CR+-CreditRisk+
  - Basel

- The three portfolios have a $66.3B total exposure each, made up of the following:
  - A High credit quality, diversified (500 names)
  - B High credit, concentrated (100 names)
  - C Low credit, diversified (500 names)
Comparative Study Continued...

Table: 1 Year Horizon, 99% Confidence

<table>
<thead>
<tr>
<th>Assuming 0 Correlation</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>CM</td>
<td>777</td>
<td>2093</td>
<td>1989</td>
</tr>
<tr>
<td>CR+</td>
<td>789</td>
<td>2020</td>
<td>2074</td>
</tr>
<tr>
<td>Basel</td>
<td>5304</td>
<td>5304</td>
<td>5304</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Assuming Correlation</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>CM</td>
<td>2264</td>
<td>2941</td>
<td>11436</td>
</tr>
<tr>
<td>CR+</td>
<td>1638</td>
<td>2574</td>
<td>10000</td>
</tr>
<tr>
<td>Basel</td>
<td>5304</td>
<td>5304</td>
<td>5304</td>
</tr>
</tbody>
</table>

When going from a 0 correlation to correlated model note that:
- **Models are fairly consistent** (between CM and CR+ when 0 correlation is assumed).
- **Correlations increase credit risk**
- **There is a higher discrepancy between models**

When going from a 0 correlation to correlated model note that:

- Models are fairly consistent (between CM and CR+ when 0 correlation is assumed).
- Correlations increase credit risk
- There is a higher discrepancy between models
Example (23-7: FRM Exam 1999)

Which of the following is used to estimate the probability of default for a firm in the KMV model?

I Historical probability of default based on the credit rating of the firm (KMV have a method to assign a rating to the firm if unrated).

II Stock price volatility.

III The book value of the firms equity

IV The market value of the firms equity

V The book value of the firms debt

VI The market value of the firms debt

a) I
b) II, IV and V
c) II, III, VI
d) VI only
Solution (23-7: FRM Exam 1999)

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V. The book value of the firms debt

VI. The market value of the firms debt

a) I
b) II, IV and V
c) II, III, VI
d) VI only
Example (23-8: FRM Exam 1999)

J.P. Morgan’s CreditMetrics uses which of the following to estimate default correlations?

I. CreditMetrics does not estimate default correlations; it assumes zero correlations between defaults.

II. Correlations of equity returns.

III. Correlations between changes in corporate bond spreads to treasury.

IV. Historical correlation of corporate bond defaults.
Solution 23-8: FRM Exam 1999

Solution (23-8: FRM Exam 1999)

J.P. Morgans CreditMetrics uses which of the following to estimate default correlations?

I CreditMetrics does not estimate default correlations; it assumes zero correlations between defaults.

II Correlations of equity returns

III Correlations between changes in corporate bond spreads to treasury

IV Historical correlation of corporate bond defaults
Example (23-9: FRM Exam 1998)

J.P. Morgan’s CreditMetrics uses which of the following to estimate default correlations?

a) Bond spreads to treasury.
b) History of loan defaults.
c) Assumes zero correlations and simulates defaults.
d) None of the above.
Solution (23-9: FRM Exam 1998)

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a) Bond spreads to treasury.
b) History of loan defaults.
c) Assumes zero correlations and simulates defaults.
d) None of the above.
Example 23-10: FRM Exam 2000

Example (23-10: FRM Exam 2000)

- The KMV credit risk model generates an estimated default frequency (EDF) based on the distance between the current value of the assets and the book value of the liabilities.
- Suppose that the current value of a firm’s assets and the book value of its liabilities are $500M and 300M, respectively.
- Assume that the standard deviation of returns on the assets is $100M, and that the returns of the assets are normally distributed.
- Assuming a standard Merton Model, what is the approximate default frequency (EDF) for this firm?
  1. 0.010
  2. 0.015
  3. 0.020
  4. 0.030
Solution (23-10: FRM Exam 2000)

→ Assuming a standard Merton Model, what is the approximate default frequency (EDF) for this firm?

1. 0.010
2. 0.015
3. 0.020
4. 0.030

Why?

- Distance from default is calculated as:

\[
\frac{A - K}{\sigma} = 2.
\]

- The default probability is then 0.023 (using a Gaussian model).
Example (23-11: FRM Exam 2000)

Which one of the following statements regarding credit risk models is MOST correct?

1.) The CreditRisk+ model decomposes all the instruments by their exposure and assesses the effect of movements in risk factors on the distribution of potential exposure.

2.) The CreditMetrics model provides a quick analytical solution to the distribution of credit losses with minimal data input.

3.) The KMV model requires the historical probability of default based on the credit rating of the firm.

4.) The CreditPortfolioView (McKinsey) model conditions the default rate on the state of the economy.
Which one of the following statements regarding credit risk models is MOST correct?

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4.) The CreditPortfolioView (McKinsey) model conditions the default rate on the state of the economy.
Solution (23-11: FRM Exam 2000 Continued...)

Which one of the following statements regarding credit risk models is MOST correct?

4.) The CreditPortfolioView (McKinsey) model conditions the default rate on the state of the economy.

Why?

- The CreditRisk+ assumes fixed exposure.
- CM is simulation
- KMV uses the current stock price