

The Business Case

I am holding a 15-year treasury zero on a \$1,000 notional. What is my VaR?

(Note: this instrument does not exist)

The Model

- No default risk
- 1 day VaR
- 95% confidence
- log-normally distributed yield
- Present value: \$370.
- The yield changed today by 0.01

The Calculation

$$\begin{aligned} 0.05 &= \text{Prob} \{ N e^{-r_0} - N e^{-r} \geq \mathbf{VaR} \} \\ &= \text{Prob} \{ N e^{-r_0} (1 - e^{r_0-r}) \geq \mathbf{VaR} \} \\ &= \text{Prob} \{ 1 - e^{r_0-r} \geq \mathbf{VaR} / 370 \} \\ &= \text{Prob} \{ r - r_0 \geq -\log(1 - \mathbf{VaR} / 370) \} \end{aligned}$$

Hence,

$$\begin{aligned} \log(1 - \mathbf{VaR} / 370) &= -1.65 \cdot \sigma \\ &= -0.0165, \end{aligned}$$

The Result

$$\begin{aligned}\mathbf{VaR} &= 370 \cdot (1 - e^{-0.0165}) \\ &= 370 \cdot .016365 \\ &= \$6.05\end{aligned}$$

Bond – VaR

Zero coupon bond with notional N_0 . Its price is

$$P(R(t)) = N_0 e^{-R(t)},$$

where $R(t)$ is the one-year interest rate.

$$R = R_0 e^{\zeta},$$

where R_0 is today's interest rate and ζ is a normal distributed random variable with mean zero and variance σ^2 (the daily volatility of the interest rate).

R_0 and R today's and tomorrow's interest rates, respectively.

α -Var is defined through

$$\text{Prob} \{ N_0 e^{-R_0} - N_0 e^{-R} > P_\alpha \} = \alpha.$$

We solve this by setting $\lambda_\alpha = P_\alpha N_0^{-1} e^{R_0}$:

$$\begin{aligned}\alpha &= \text{Prob} \left\{ 1 - e^{R_0(1-e^\zeta)} > \lambda_\alpha \right\} \\ &= \text{Prob} \left\{ e^\zeta > 1 - \frac{1}{R_0} \log(1 - \lambda_\alpha) \right\} \\ &= \text{Prob} \left\{ \zeta > \log \left(1 - \frac{1}{R_0} \log(1 - \lambda_\alpha) \right) \right\}.\end{aligned}$$

As ζ is a $N(0, \sigma^2)$ random variable, we obtain that

$$\log \left(1 - \frac{1}{R_0} \log(1 - \lambda_\alpha) \right) = \sigma q_\alpha,$$

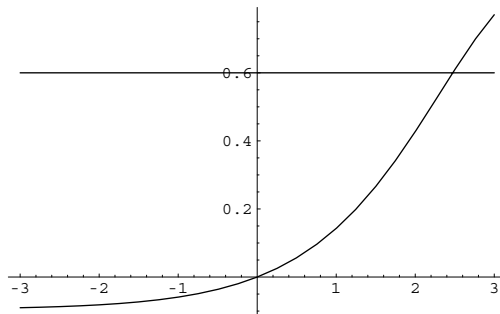
where q_α is the α -quantile of the standard normal distribution, that is

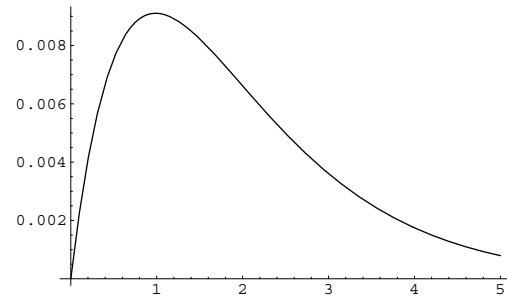
$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{q_\alpha}^{\infty} e^{-y^2/2} dy.$$

With the above definition of λ_α , we can solve for P_α to obtain

$$P_\alpha = N_0 e^{-R_0} \left(1 - e^{R_0(1-e^{\sigma q_\alpha})} \right).$$

It is important to note that this exact calculation is possible because the increment in the value of the bond $P(0) - P(1)$ is an increasing function of the random variable ζ . And this allows us to invert the relation between both of them.





VaR in terms of the interest rate R_0

An approximation Approach

Taylor series:

$$P(R) \approx P(R_0) + \underbrace{\frac{\partial P}{\partial R} \Big|_{R=R_0}}_{\delta} (R - R_0) + \frac{1}{2} \underbrace{\frac{\partial^2 P}{\partial R^2} \Big|_{R=R_0}}_{\Gamma} (R - R_0)^2$$

Linear approx.

$$\delta = \frac{\partial(N_0 e^{-R})}{\partial R} \Big|_{R=R_0} = -N_0 e^{-R_0}.$$

The P&L:

$$f(\zeta) = P(R_0) - P(R) = -\delta(R - R_0) = N_0 R_0 e^{-R_0} (e^\zeta - 1).$$

And, as is an increasing function of ζ ,

$$P_\alpha = N_0 e^{-R_0} R_0 (e^{\sigma q_\alpha} - 1).$$

Quadratic attempt.

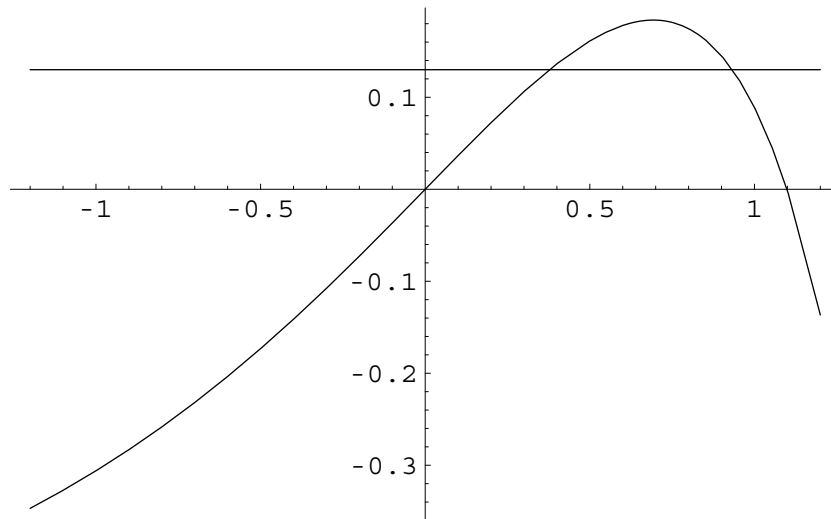
$$\Gamma = \frac{1}{2} \frac{\partial^2(N_0 e^{-R})}{\partial R^2} \Big|_{R=R_0} = N_0 e^{-R_0}.$$

P&L:

$$g(\zeta) = N_0 e^{-R_0} R_0 \left[e^\zeta - 1 - \frac{R_0}{2} (e^\zeta - 1)^2 \right].$$

And this not an increasing function any more. In fact, it attains a maximum at

$$\zeta = \log(1 + 1/R_0).$$



Picture of the function $g(\zeta)$.

RiskMetrics

The principal assumption of RiskMetrics methodology is that the log-returns of the underlying are small, and so, we can estimate the difference between two consecutive values keeping only the first term of the corresponding Taylor series. In our case, that means

$$R - R_0 = R_0 e^\zeta - R_0 \sim R_0 \zeta.$$

With this assumption, we can perform the same calculation as before:

$$\begin{aligned} P(R_0) - P(R) &= -\delta(R - R_0) - \Gamma(R - R_0)^2 - \dots \\ &= -\delta R_0(\zeta + \zeta^2 + \dots) - \Gamma R_0^2(\zeta + \dots)^2 + \dots \end{aligned}$$

For the δ -approximation, we should only keep the first order terms in ζ , that is,

$$P(R_0) - P(R) = -\delta\zeta = N_0 e^{-R_0} R_0 \zeta.$$

Again, this is an increasing function of ζ , and the VaR is easily calculated:

$$\alpha - \text{VaR}_\delta = N_0 e^{-R_0} R_0 \sigma q_\alpha.$$

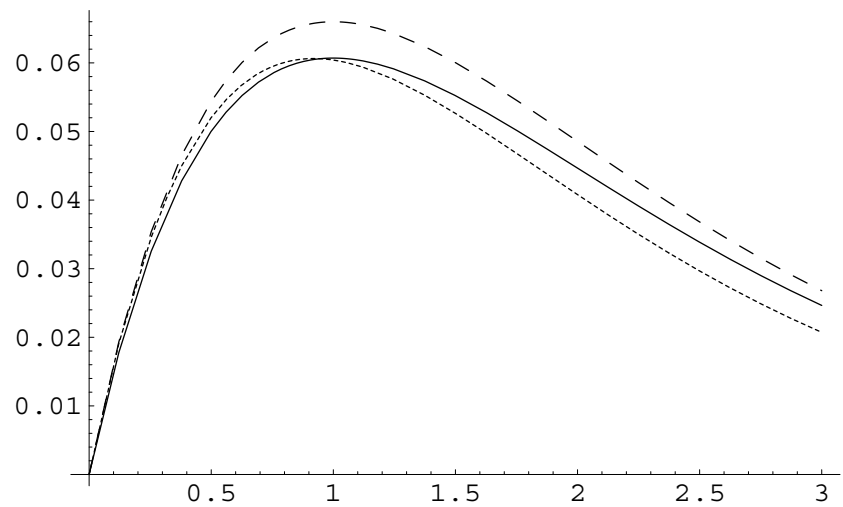
For the second order approximation, we should keep all the terms up to second order in ζ , that is,

$$\begin{aligned} P(R_0) - P(R) &= -\delta R_0 \zeta - \delta R_0 \zeta^2 - \Gamma R_0^2 \zeta^2 \\ &= N_0 e^{-R_0} R_0 \left[\zeta + \zeta^2 \left(1 - \frac{R_0}{2} \right) \right]. \end{aligned}$$

And the $\alpha - \text{Var}_{\delta-\Gamma}, P_\alpha$ is now calculated as:

$$\alpha = \text{Prob} \left(\zeta + \left(1 - \frac{R_0}{2} \right) \zeta^2 > \frac{P_\alpha}{N_0 e^{-R_0} R_0} \right)$$

Again, this can be solved numerically for P_α .



Comparison of the different VaRs obtained. In continuous line, the RiskMetrics approach. In small-dotted line, the exact value. In big-dotted, the first order approximation.

Greeks

They will account for portfolio changes due to small market moves.

They are basic to risk management since full portfolio re-evaluation may be too costly or too inaccurate.

They are used for institution wide risk management, as well as for front office risk management.

Delta

It is the hedge ratio for an infinitely small move:

$$\delta_i = \frac{\partial \Pi}{\partial r_i}.$$

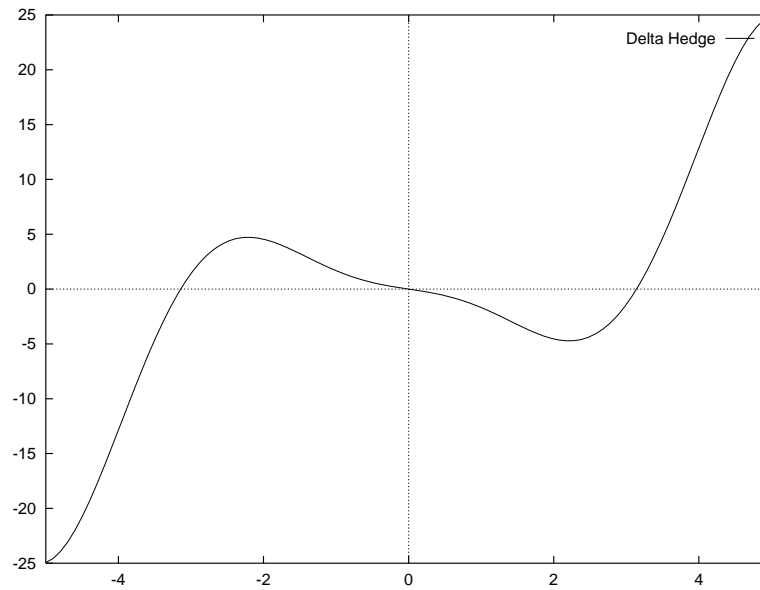
It is useful for

- (almost) linear portfolios
- medium-length options close to the money.

Useless for almost everything else.

An example

Consider an option with a P&L given by



This instrument is short delta for small terms, long delta for large market moves.

The choice of units

Hedging practices encourage the use of the cash delta; this is the sensitivity with respect to the traded underlying.

For other purposes, other risk factor sensitivities may be more interesting.

For example, for risk management of european options, the forward delta may be more appropriate.

Gamma

The second derivative

Provides a longer term correction to the delta.

For example:

- For an at-the-money option, gamma is large close to expiration.
- For an out-the-money option, gamma is large far from expiration.

Duration

$$\begin{aligned} D &= \sum_i t_i \cdot PV \frac{\text{payment}_i}{\text{Total payments}}. \\ &= \frac{1}{V} \sum_i t_i \cdot p_i \cdot e^{-r_i t_i} \end{aligned}$$

where

$$V = \sum_i p_i \cdot e^{-r_i t_i}.$$

Assume a flat yield curve, i.e. $r_i = r$ for all i . Then,

$$D = -\frac{1}{V} \cdot \frac{\partial V}{\partial r}$$

More generally, the duration expresses the sensitivity with respect to parallel shifts of the yield curve:

$$D = \frac{1}{V} \left. \frac{\partial V}{\partial \varepsilon} \right|_{\varepsilon=0} (r_1 + \varepsilon, \dots, r_n + \varepsilon).$$

Vega

Sensitivity with respect to the volatility.

It assumes volatility is not constant. Typically, a function of strike price, expiration.

Often, volatility is measured with respect to term and time, each with discrete observations. The result is a family of volatilities, arranged as a volatility matrix.

The vega is the sensitivity with respect to each.

Since volatility is not observed, it accounts both for future volatility moves or volatility calibration risk.

Implied volatility

For hedging purposes, volatilities are usually not measured, but implied from liquid option prices (i.e., caps, swaptions, calls, etc.)

The implied volatility is usually dependent of the particular instrument that it is calibrated to. Often, they show a geometric shape similar to a smile. It shows the market's view of its future evolution as opposed to the underlying theoretical model used for valuation (usually Black-Scholes for equity, Black for caps, Hull-White for interest rate derivatives).

Delta Normal VaR

Approximate

$$\Pi(t) - \Pi(0) \approx \left. \frac{\partial \Pi}{\partial t} \right|_{t=0} + \sum_{i=1}^n \delta_i \cdot [S_i(t) - S_i(0)],$$

where

$$\delta_i = \left. \frac{\partial \Pi}{\partial S_i} \right|_{t=0}.$$

VaR is then given by

$$\text{Prob} \left\{ \sum_{i=1}^n \delta_i \cdot [S_i(t) - S_i(0)] < -\mathbf{VaR} \right\} = 0.05.$$

If $S_i(t) - S_i(0)$ is normally distributed, with 0 mean and variance/covariance matrix given by \mathbb{W} , this is equivalent to

$$\int_{\delta \cdot x < -\mathbf{VaR}} e^{-x^\dagger \mathbb{W}^{-1} x / 2} \frac{dx}{\sqrt{\det(2\pi \mathbb{W})}} = 0.05.$$

We now introduce the Cholesky decomposition of $\mathbb{W} = \mathbb{H} \mathbb{H}^\dagger$, and change variables

$$x \mathbb{H}^{-1} = y,$$

to obtain

$$\int_{\delta \mathbb{H} y^\dagger < -\mathbf{VaR}} e^{-|y|^2 / 2} \frac{dy}{(2\pi)^{n/2}} = 0.05.$$

Let \mathbb{A} be the rotation that sends $\delta \mathbb{H}$ into $(|\delta \mathbb{H}|, 0, \dots, 0)$, and change variables $y = z \mathbb{A}$, to obtain

$$\int_{|\delta \mathbb{H}| z_1 < -\mathbf{VaR}} e^{-|z|^2 / 2} \frac{dz}{(2\pi)^{n/2}} = 0.05.$$

Since

$$\int_{|\delta\mathbb{H}|z_1 < -\mathbf{VaR}} e^{-|z|^2/2} \frac{dz}{(2\pi)^{n/2}} = \int_{-\infty}^{-\mathbf{VaR}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}},$$

we conclude that

$$\begin{aligned}\mathbf{VaR} &= -z\sqrt{|\delta\mathbb{H}|} \\ &= -z\sqrt{\delta^T \cdot \mathbb{V} \cdot \delta},\end{aligned}$$

where \mathbb{V} is the variance/covariance matrix, and z is the 95% percentile of the univariate normal distribution.

- Simple
- Inaccurate for non-linear instruments.

Example

Let's attempt to recreate the situation of a Repo contract (say like in Orange County).

We would have a portfolio $\Pi(r_1, \dots, r_n)$ of interest rate instruments, depending therefore on yield rates r_i , for i ranging from 1 (overnight lending rate) to 14 (the 30 year rate).

The yield curve will have a covariance matrix \mathbb{W} that measures the market volatility with respect to motions of the interest rate curve.

We can find the principal components of \mathbb{W} , which will give us the directions of movement in the market that will tend to be more pronounced.

In doing this, we may find that parallel shifts of the yield curve will be most likely, with a 60% chance. Tilts will be next, with a 25% chance.

Our portfolio is perfectly hedged against parallel shifts; it is exposed to tilts. In

other words, if we map the risk factors r_i into the principal components, and keep only the first two ones (then most likely ones), we may find

$$\delta = (0, 50\%),$$

expressed in percentage points.

It would be a mistake to think that the portfolio is risk-free because it is insensitive to parallel shifts. Similarly, it would be a mistake to think that the portfolio is risk-free because if we hold to maturity, it will make money with certainty.

In fact, Delta-Normal-VaR will, in our situation, yield

$$1.65 \cdot \sqrt{\delta^\top \mathbb{V} \delta} \approx 40.$$

This means that we should expect our portfolio to lose 40% of its value once a month.

Duration

$$\begin{aligned} D &= \sum_i t_i \cdot PV \frac{\text{payment}_i}{\text{Total payments}}. \\ &= \frac{1}{V} \sum_i t_i \cdot p_i \cdot e^{-r_i t_i} \end{aligned}$$

where

$$V = \sum_i p_i \cdot e^{-r_i t_i}.$$

Assume a flat yield curve, i.e. $r_i = r$ for all i . Then,

$$D = -\frac{1}{V} \cdot \frac{\partial V}{\partial r}$$

Assignment

Compute the VaR for the desk1 bond using the following methodologies:

- Duration matching
- δn VaR
- MonteCarlo