Mathematics and Finance: past, present and future

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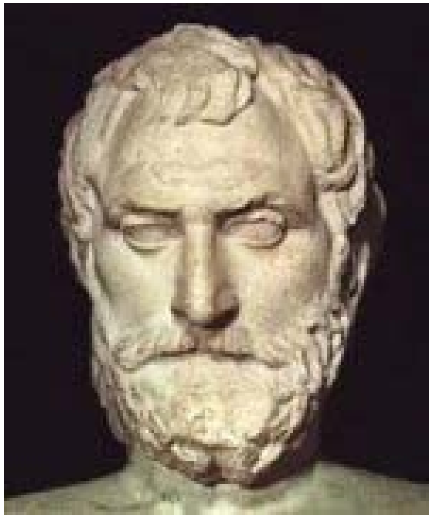

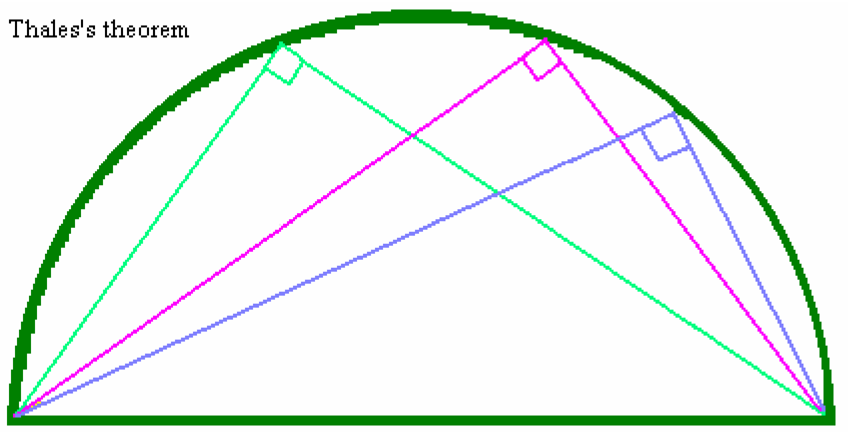

Mathematics and finance

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$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

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- October 19, 1987: New York
- Summer, 1998:
  - The Russian default; Clinton - Lewinsky

Overview of Credit Markets

Bond: issued by A

A promised to pay back the principal and all interest

A owes money to B

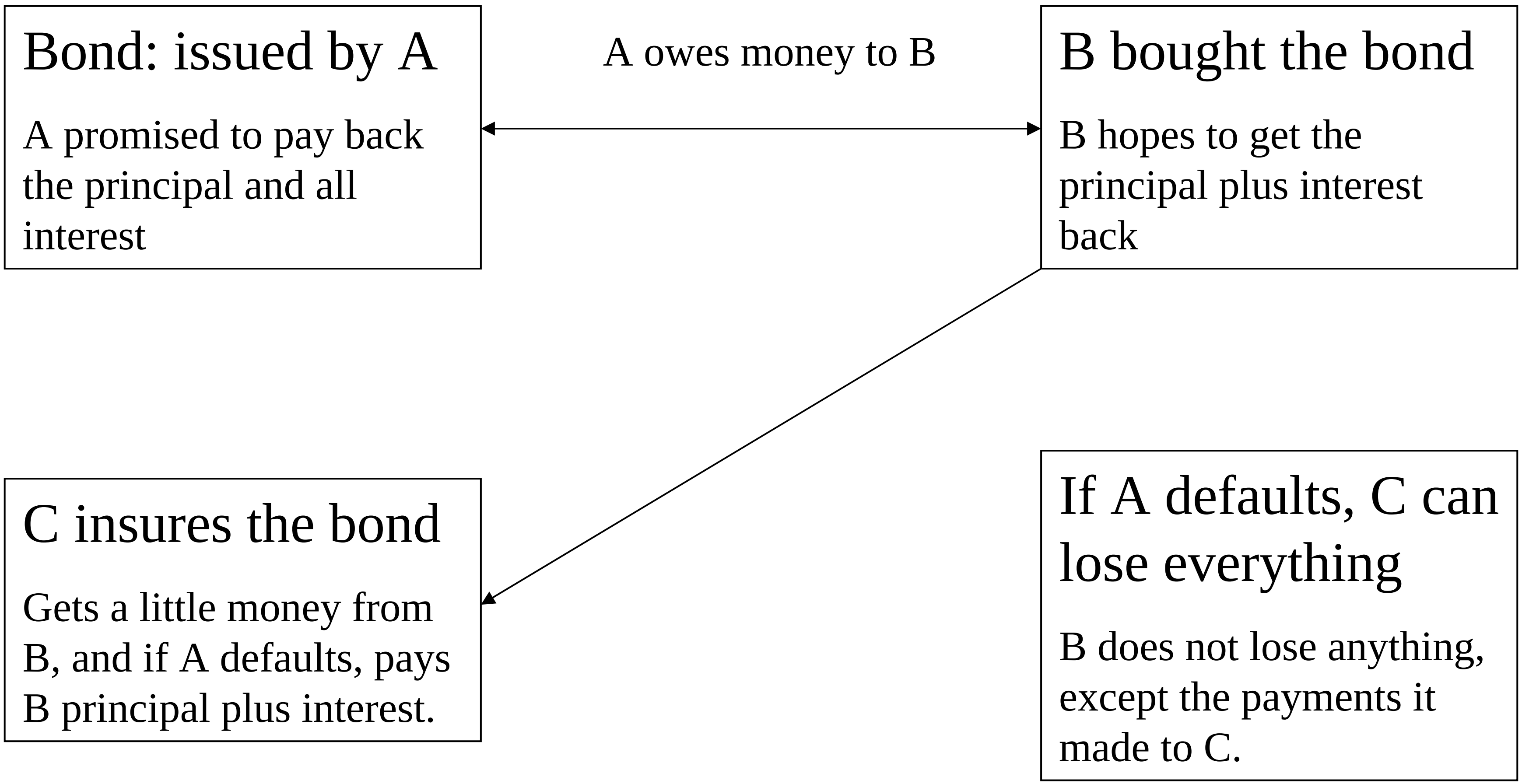
B bought the bond

B hopes to get the principal plus interest back

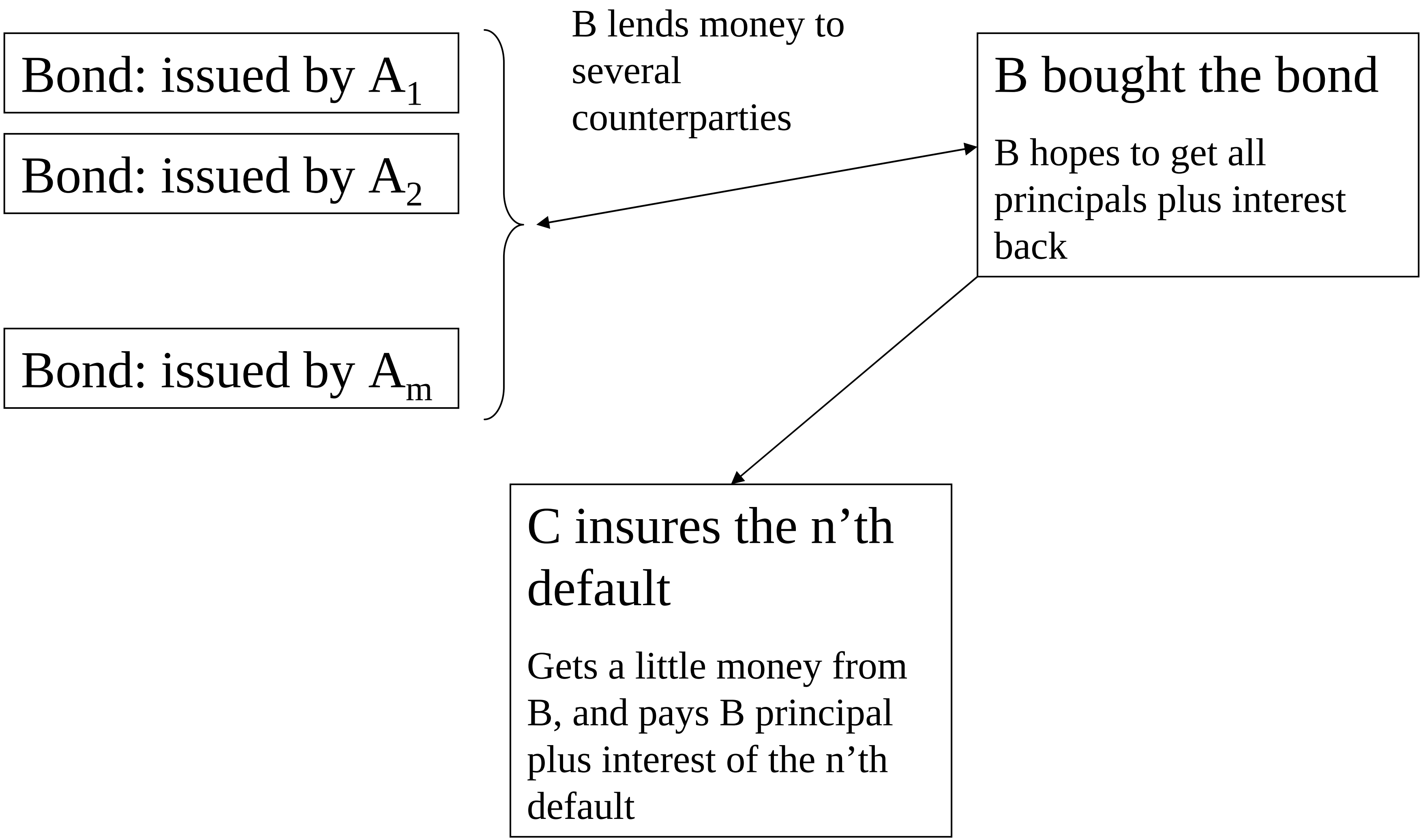
If A defaults, B can lose everything

That’s why B pays little less for the bond that if, say, it had been issued by the government

Credit protection



n’th to default swap



Collateralized Debt Obligation (CDO)

Bond: issued by A_1

Bond: issued by A_2

Bond: issued by A_m

B lends money to several counterparties

B bought the bond

B hopes to get all principals plus interest back

C_1 insures the first n_1 defaults

C_2 insures the next n_2 defaults

Etc…

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Etc…

C_1 receives a high yield

C_2 receives a smaller yield

Etc…

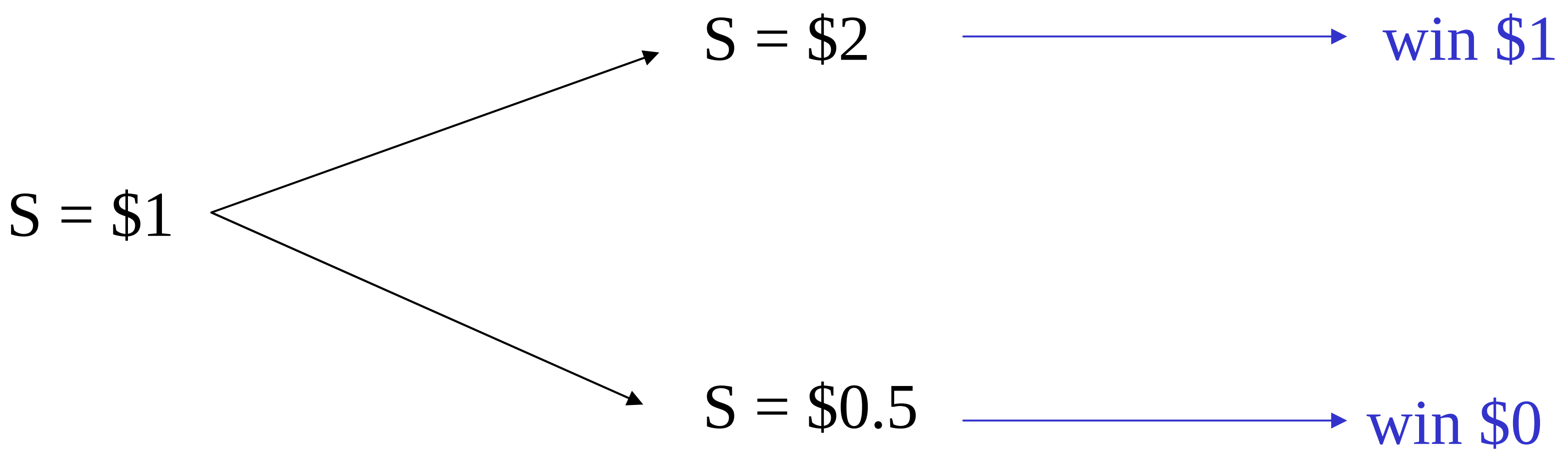
Pricing credit:

Market participants need to determine the credit premium (yield, or spread) at the time of sale.

Whether there is a default event or not, the price of a credit derivative (such as a CDS) will evolve (up or down) as the credit worthiness of the counter-parties changes. It can also be affected by market prices.

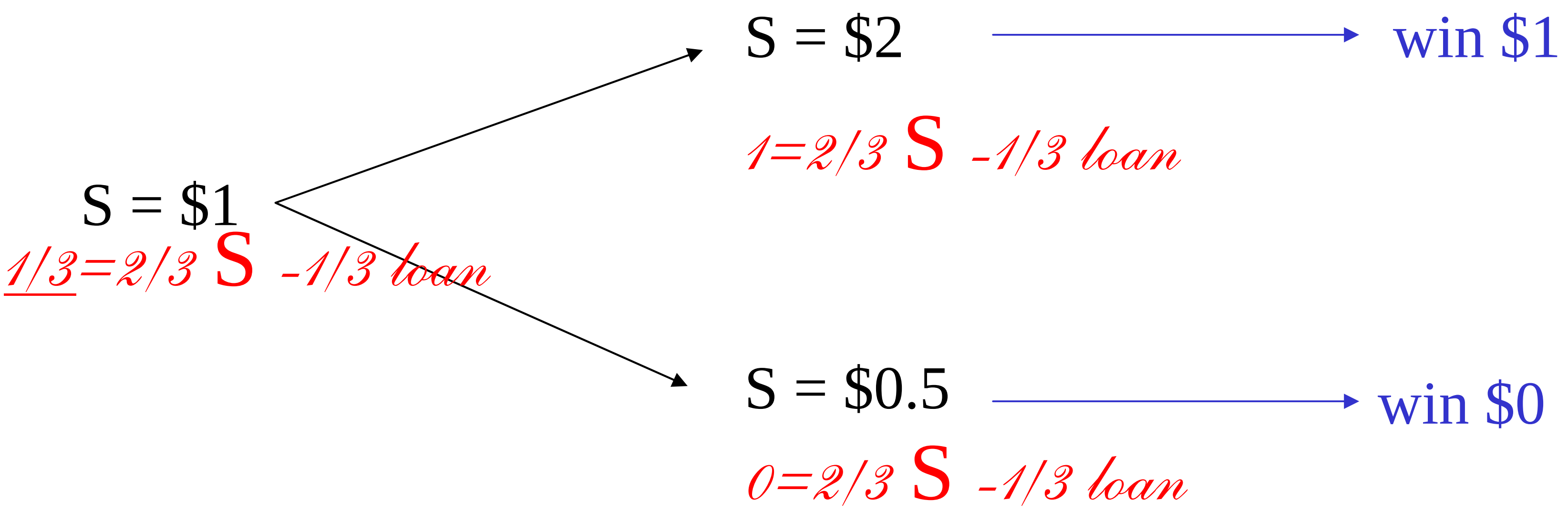
Institutions need to monitor the value of their credit derivatives on a daily basis (whether there is default or not), since substantial losses can occur even when there is no bankruptcy.

Pricing fundamentals



Assume you get $1 if S raises to $2. How much is this worth?

Pricing fundamentals



Assume you get $1 if S raises to $2. How much is this worth?

Structural Framework

Basic assumptions.

- Firm i default as soon as $V_i(t) < D_i$. Where V_i is the firm assets value.
- $V_i(t)$ follow log-normal processes with constant drift and volatility.
- Interest rate is constant, r.

Remarks:

- τ_i- time of default of firm i, then:
  $P(\tau_i < t) = P(Z_i(t) < \ln D_i)$
  $Z_i(t) = \min_{0 \le s \le t} \ln V_i(s)$.

- The probability of at least j default before t:

$$P(j,t) = P(\tau^j \leq t) \tag{1}$$
$$= [1 - P(Z_1(t) > D_1, ..., Z_i(t) > D_i, ..., Z_n(t) > D_n)]$$
$$- P(\tau^j \leq t, \tau^{j+1} > t)$$

- $\tau^j \sim$ time when j-default occurs; its density is denoted by $f_j(.)$. The probability of exactly j default before t ($\pi_t(j)$) is:

$$\pi_t(j) = P(\tau^j \leq t, \tau^{j+1} > t) \tag{2}$$
$$= \sum_{i_1 \neq i_j = 1}^{\frac{k!}{j!(k-j)!}} P(Z_1(t) > D_1, ..., Z_{i_1}(t) \leq D_{i_1}, ..., Z_{i_j}(t) \leq D_{i_j}, ..., Z_n(t) > D_n)$$

- Notice that the multivariate density of (τ^j, V) can be computed by using the distribution of (Z,V).

The PDE

Let us assume: $X(t) = \mu \cdot t + \Sigma \cdot w(t)$, $t \geq 0$, where $\mu_{n\times 1}$ and $\sigma_{n\times n}$ are constants. Denote:

$$P(X(t) \in dx, \underline{X}(t) \geq m) = p(x, t, m, \mu, \Sigma) \prod_{i=1}^{n} dx_i$$

where $x_i > m_i \, m_i \leq 0; \forall i$.

Then p (joint density) should satisfy the Fokker-Planck PDE, with initial and absorbing boundary conditions::

$$\begin{cases} \dfrac{\partial p}{\partial t} = \sum\limits_{i=1}^{n} \mu_i \cdot \dfrac{\partial p}{\partial x_i} + \sum\limits_{i,j=1}^{n} \dfrac{\sigma_{ij}}{2} \cdot \dfrac{\partial^2 p}{\partial x_i \partial x_j} \\ p(x, t=0) = \prod_{i=1}^{n} \delta(x_i) \\ p(x_i = m_i, t) = 0 \ , \quad i = 1, ..., n \end{cases} \tag{3}$$

Particular case, N=2. (See Rebholz 1998)

$$
\begin{aligned}
&P(Z_1(t) < m_1, Z_2(t) < m_2) \\
&= e^{a_1 m_1 + a_2 m_2 + bt} \cdot \frac{2}{\alpha t} \cdot \sum_{n=1}^{\infty} \sin(\frac{n\pi\theta^*(m_1)}{\alpha}) \\
&\cdot e^{\frac{-g(m_1, m_2)}{2t}} \cdot \int_0^{\alpha} \sin(\frac{n\pi\theta(m_1)}{\alpha}) \cdot g_n(m_1, m_2) d\theta
\end{aligned} \tag{4}
$$

Where $g_n(m_1, m_2)$ is an integral of Bessel functions.

Problems in higher dimensions.

One would need to be able to solve this PDE in dimension 100.

One would need to be able to know he dependence of the solution on errors in the calculation of the correlation matrices: specifically

Question If p_1 and p_2 are two solution to the PDE, arising from correlation matrices Σ_1 and Σ_2, is it true that

$$\|p_1 - p_2\|_1 \leq C\|\sigma_1 - \Sigma_2\|_2$$

Pricing n^{th} to default CDS.

Extra Assumptions

1 - Principals and the expected recovery rates associated with all the underlying entities are the same, $L = 1$, R.

2 - In the event of and an j^{th} default occurring the sellers pays the notional principal times $(1 - R)$. We could also assume $R(V_{i_1}(\tau^j), ..., V_{i_j}(\tau^j))$, we know the multivariate distribution from slide 6.

3 - We assume only one payment (from the buyer) at the beginning of the contract.

Proposition: The present value (t) of the expected payoff of a j^{th} to default CDS is:

$$E_t^Q \left[(1 - R) \cdot e^{-r(\tau^j - t)} \cdot 1_{\{\tau^j < T\}} \right] \tag{5}$$

$$= \int_t^T (1 - R) \cdot e^{-r(s-t)} \cdot f_j(s) ds \tag{6}$$

Pricing Percent of defaults CDO.

Extra Assumptions

1 - Principals (L) associated with all the underlying names are the same.

2 - The tranche i is responsible for between $q_i\%$ and $q_{i+1}\%$ of defaults in a CDO where there are N names.

3 - The principal to which the promised payments are applied declines as defaults occur.

Proposition 5: The present value of the expected cost of defaults for this tranche is the sum of the cost of defaults for n^{th} to default CDS for values of n between $q_i\%$ and $q_{i+1}\%$.

Suppose that there is a promised percentage payment of r_i at time τ. In our case the payment is $p_i\% \cdot L \cdot r_i$ with probability $1-$ $P(q_i, \tau)$, $(p_i\% - 1\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i)$, $(p_i\% - 2\%) \cdot L \cdot r_i$ with probability $\pi_\tau(q_i + 1)$, and so on.

The expected payment is therefore:

$$\begin{aligned} & p_i \cdot r_i \cdot L \cdot [1 - P(q_i, \tau)] \\ & + \sum_{j=1}^{p_i - 1} [(p_i - j) \cdot r_i \cdot L \cdot \pi_\tau(q_i + j - 1)] \end{aligned} \tag{7}$$

So the value today, t, for the tranche i is:

$$
\begin{aligned}
&\int_t^T e^{-r\cdot(s-t)} \cdot (p_i \cdot r_i \cdot L)\, ds \\
&- \int_t^T e^{-r\cdot(s-t)} \cdot (p_i \cdot r_i \cdot L) \cdot f_{q_i}(s) ds \\
&+ \sum_{j=1}^{p_i-1} \left\{ \int_t^T e^{-r\cdot(s-t)} \cdot [(p_i - j) \cdot r_i \cdot L] \cdot f_{q_i+j-1}(s) ds \right\}
\end{aligned} \tag{8}
$$

Remark The distribution of losses $Lo(t)$ for tranche i can be obtained by noticing that $P(Lo(t) = L(t) \cdot q_i + s) = \frac{\pi_t(q_i+s)}{\sum_{l=0} \pi_t(q_i+l)}$.

n^{th} to default and m^{th} worst performances Swap:

Same Assumptions as for n^{th} to default Swap.

Proposition 4: The present value (t) of the expected payoff of a n^{th} to default and m^{th} worst performances Swap is:

$$E_t^Q[(1-R)\cdot e^{-r(\tau^j-t)}\cdot 1_{\{\tau^j<T\}}]+$$
$$E_t^Q[(m\cdot S_{i_{m+1}}(\tau^j))\cdot e^{-r(\tau^j-t)}\cdot 1_{\{\tau^j<T,S_{i_1}(\tau^j)\leq\ldots\leq S_{i_m}(\tau^j)\}}] \tag{9}$$

Where $\{S_{i_j}(t)\}_{j=1}^{M}$ is the ordered stock prices at t (increasing).